Discrete Mathematics

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Published by:

NEERAJ PUBLICATIONS

(Publishers of Educational Books)

Sales Office : 1507, 1st Floor,
Nai Sarak, Delhi-110 006
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Website: www.neerajbooks.com

Typesetting by: Competent Computers

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DISCRETE MATHEMATICS

1 PROPOSITIONAL CALCULUS

1.1 INTRODUCTION

Logic is the science which deals with the methods of reasoning. Reasoning plays a very important role in every area of knowledge, particularly in mathematics. A symbolic language has been developed over the past two centuries to express the principles of logic in precise and unambiguous terms. Logic expressed in such a language has come to be called "symbolic logic".

Symbolic logic has now become a core subject of study for every student of mathematical sciences.

Logic is the discipline which thus consists of tools of reasoning. At the elementary level, logic provides rules and techniques for determining whether a given argument is valid. Logical reasoning is used in mathematical sciences to provide the theorems, in computer science to verify the correctness of programs and to prove theorems, in the natural and physical sciences to draw conclusions from experiment, and in the social sciences and in our everyday life to solve a multitude of problems. Indeed, we are constantly using logical reasoning.

Logic is concerned with all kinds of reasonings, whether they be legal arguments or mathematical proofs or conclusions in a scientific theory based on a set of hypothesis. Because of the diversity rules of their application, these rules, called inference rules, must be stated in general terms and must be independent of any particular argument of any discipline involved.

Any collection of rules or any theory needs a language in which these rules or theory can be stated. Natural languages are not always precise enough. They are also ambiguous and as such, are not suitable for this purpose. It is therefore necessary first to develop a formal language called the *object language*. A formal language is one in which the syntax is well defined. In fact, every scientific discipline develops its own object language which consists of certain well defined terms and well specified uses of these terms.

In order to avoid ambiguity, we use symbols which have been clearly defined in the object languages. An additional reason to use symbols is that they are easy to write and to manipulate. Because of this use of symbols, the logic that we shall study is also called *symbolic logic*. Our study of object language requires the use of another language. For this purpose we choose any of the natural languages. In this case our choice is English, and so the statements about the object language will be made in English. This language will then be called our **metalanguage**. Certain inherent difficulties in this procedure could be anticipated, because we wish to study a precise language while using another language which is not so precise.

1.2 STATEMENTS

A statement is a declarative sentence which has one and only one of two possible values, called **'truth values'**. These two truth values are 'true' and 'false', denoted by the symbols T and F respectively; sometimes these are also denoted by the symbols 1 and 0. We assume that object language contains such declarative sentences. Since we allow only two possible truth values in our study, this logic is called *two valued logic.*

Declarative sentences in the object language are of two types. The first type includes those sentences which are considered to be primitive in the object language. These will be denoted by distinct symbols selected from the capital letters A, B, C, ...P, Q, ... , while declarative sentences of the second type are obtained from the primitive ones by using certain symbols, called **connectives** and certain punctuation marks, such as parantheses.

We shall now give examples of statements and show why some of them are not admissible in the object language and, hence will not be symbolized.

(i) India is a country.

(ii) New Delhi is the capital of Nepal.

(iii) This statement is true.

(iv) 1 + 101 = 110

(v) Close the door.

(vi) Nalanda is an old city.

(vii) Man will reach Mars by 2005.

Obviously the statements *(i)* and *(ii)* have truth values true and false respectively, *(iii)* is not a statement. In *(iv)* we have a statement whose truth value depends on the context e.g. if the numerals are binary digits, it is true, otherwise if they are decimal digits, it is flase. We shall be interested only in the fact that it has a truth value. In this sense *(iv)*, *(vi)*, *(vii)* are all statements.

1.3 PROPOSITIONS

Let us consider the following sentences:

1. The number 3 is a prime.
2. Every prime is a multiple of 3.
3. Every square is a rectangle.
4. Every rectangle is a square.

Each of the above sentences is a statement (declaration) which can be decisively said to be either true or false. In fact sentences 1 and 3 are true statements whereas sentences 2 and 4 are false statements. Sentences of this type are called **propositions**.

We may define proposition as follows:

"*A proposition is a statement which, in a given context, can be said to be either true or false, but not both.*"

It should be noted that not all sentences are propositions. For example, consider the following sentences:

1. Take a triangle ABC.
2. $xy = yx$.

We note that the first of these sentences is not at all a statement; as such, it is not a proposition. The second sentence is a statement, but we cannot decisively say whether it is true or not, unless we know what x and y are. As such, it is also not a proposition. However, if the context tells that x and y are, say, real numbers, then it becomes a proposition.

Propositions are usually denoted by small letters such as $p, q, r, s, ...$. The truth or the falsity of a proposition is called its ***truth value***. If a proposition is true, we will indicate its truth value by the symbol T and if it is false by the symbol F.

For example, if we denote the proposition "The number 3 is a prime number" by p, then the truth value of p is T. Similarly, if we denote the proposition "Every rectangle is a square" by q, then the truth value of q is F.

SOLVED EXAMPLE

Example 1. Write down up, where *p* is

(a) $A - 10 \neq 10$

(b) $n > 6$ for every $n \in N$

(c) Mostly girls top in the IAS examination

Sol. *(a) Given p:* $A - 10 \neq 10$

$\therefore \quad \sim p : A - 10 = 10$

(b) Given p: $n > 6$ for every $n \in N$

$\therefore \quad \sim p : n \ngtr 6$ for at least one $n \in N$

We can write this as 'n is not greater than 6 for every $n \in N$', or, there is at least one n such that $n \in N$ and for which $n \leq 6$.'

(c) Given p: Mostly girls top in the IAS examination.

$\therefore$ $\sim p$: Mostly girls do not top in IAS examination.

1.4 COMPOUND PROPOSITIONS

New propositions are often formed (constructed) by starting with given propositions with the aid of words or phrases like '*not', 'and', 'if...then',* and *'if and only if'*. Such words or phrases are called *connectives.* The new propositions obtained by the use of connectives are called *compound propositions.* The original propositions from which a compound proposition is formed are called the *components* of the compound proposition.

In mathematics, the letters *x, y, z*, . . . often denote variables that can be replaced by real numbers, and these variables can be combined with the familiar operations +, ×, – and ÷. In logic, the letters *p, q, r,* ... denote **propositional variables,** that is, variables that can be replaced by statements. Thus we can write *p*: The sun is shining today. *q*: It is cold. Statements or propositional variables can be combined by logical connectives to obtain **compound statements.** For example, we may combine the preceding statements by the connective *and* to form the compound statement *p* and *q*: The sun is shining *and* it is cold. The truth value of a compound statement depends only on the truth values of the statements being combined and on the types of connectives being used. We shall now look at the most important connectives.

1.4.1 Negation

A proposition obtained by inserting the word '*not*' at an appropriate place in a given proposition is called the *negation* of the given proposition. The negation of a proposition *p* is denoted by $\sim p$ (read '*not p*'), the symbol '~' denoting the word 'not'.

For example, let the proposition "2 is a prime number" be denoted by *p*; that is,

p: 2 is a prime number.

Then the negation of *p* is "2 is not a prime number"; that is,

$\sim p$: 2 is not a prime number.

Similarly, let

p: Every rectangle is a square.

Then

$\sim p$: Not every rectangle is a square.

It is obvious that, for any proposition *p*, if *p* is true , *then* $\sim p$ *is false, and if p is false, then* $\sim p$ *is true.* This observation can be recorded in the form of a **truth table** as given below:

Table 1.1*: Truth Table for Negation*

p	*~p*
T	F
F	T

The first column of the above table gives the possible truth values of *p* and the second column gives the corresponding truth values of $\sim p$.

SOLVED EXAMPLE

Example 2. Write down the disjunction of following propositions, and give its truth value:

(1) 5 + 8 = 9

(2) Asha is an architect.

Sol. Let 5 + 9 = 8 be statement *p*, and Asha is an architect be statement *q.* The disjunction is '5 + 8 = 9 or Asha is an architect.' Now, 5 + 8 = 9 is always false. Therefore, the truth vlaue of the disjunction depends on the truth value of 'Asha is an architect.' If it is true then we have *p* or *q*

$\Rightarrow$ (False) or (True)

$\Rightarrow$ True **Ans.**

However, if Asha is not an architect, then we have *p* or *q*

$\Rightarrow$ (False) or (False)

$\Rightarrow$ False **Ans.**

■ The Exclusive OR Connective (⊕)

The disjunction '$p \vee q$' is true not only when either *p* or *q* is true, but also when both *p* and *q* are true. This is called inclusive or it may happen that we want to ensure that only one of them should be true. In such a situation the connective 'inclusive OR' is used.

Definition of Exclusive OR

The exclusive disjunction of two propositions *p* and *q* is the statement 'Either *p* is true or *q* is true, but both are not true.' This type of disjunction is denoted by $p \oplus q$.

For example, if p is '5 + 10 = 15' and q is the statement, 'Mohini is clever,' then $p \oplus q$ is the statement.

'Either 5 + 10 or Mohini is clever.' This will be true only if Mohini is not clever, because 5 + 10 = 15 is always true.

■ **Truth Table for Exclusive OR**

p	***q***	$p \oplus q$
T	T	F
T	F	T
F	T	T
F	F	F

1.4.2 Conjunction

A compound proposition obtained by combining two given propositions by inserting the word *'and'* in between them is called the *conjunction* of the given propositions.

The conjunction of two propositions p and q is denoted by $p \wedge q$ (read "p and q"), the symbol "$\wedge$" denoting the word '*and*'.

For example, let

p: 2 is a prime number.

q: 6 is a multiple of 3

Then

$p \wedge q$: 2 is a prime number and 6 is a multiple of 3.

The following *rule* is adopted in deciding the truth value of a conjunction:

The *conjunction $p \wedge q$ is true only when p is true and q is true; in all other cases it is false.*

The above rule can be expressed more explicitly in the form of a truth table as given below:

Table 1.2: *Truth Table for Conjunction*

p	***q***	$p \wedge q$
T	T	T
T	F	F
F	T	F
F	F	F

In the above table the last column gives the truth values of $p \wedge q$ for all possible combinations of the truth values of p and q given in the first two columns.

As an illustration, let us consider the following propositions:

p: $\sqrt{2}$ is an irrational number.

q: 9 is a prime number.

r: All triangles are equilateral.

Then

$p \wedge q$: $\sqrt{2}$ is an irrational number and 9 is a prime number.

$q \wedge r$: 9 is a prime number and all triangles are equilateral.

$r \wedge p$: All triangles are equilateral and $\sqrt{2}$ is an irrational number.

Here, p is true and q and r are false.

Since p is true and q is false, $p \wedge q$ is false; since both q and r are false, $q \wedge r$ is false; since r is false and p is true, $r \wedge p$ is false.

If we consider one more proposition.

s: All squares are rectangles,

then

$p \wedge s$: $\sqrt{2}$ is an irrational number and all squares are rectangles.

Here both p and s are true. Hence $p \wedge s$ is true.

SOLVED EXAMPLE

Example 3. Find out the set of the real numbers x for which the truth value of $p \wedge q$ is T. Take p: $x > -3$ and q: $x + 5 \neq 12$.

Sol. The proposition p will have a truth value only for $x \in]-3, \infty[$ and $x \neq 7$. This means the set of real numbers of x for which $p \wedge q$ is true, is given by

$x \in]-3, >[\cup] 7, \infty[$ **Ans.**

1.4.3 Disjunction

A compound proposition obtained by combining two given propositions by inserting the word '*or*' in between them is called the *disjunction* of the given propositions.

The disjunction of two propositions p and q is denoted by $p \vee q$ (read "p or q"), the symbol $\vee$ denoting the word '*or*'.

For example, let

p: 2 is a prime number.

q: 6 is an odd number.