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ADVANCED CALCULUS

By:
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Sample Preview of the Solved Sample Question Papers

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QUESTION PAPER

(June – 2018)

(Solved)

ADVANCED CALCULUS

Time: 2 Hours] *[Maximum Marks: 50*
(Weightage: 70%)

Note: *Attempt **four** questions in all. Question No. **1** is **compulsory**. Use of calculator is not allowed.*

Q. 1. State whether the following statements are *True* or *False*. Justify your answer.

***(a)* If $f(x) = 1/x$ and $g(x) = \tan x$, then the domain of $f + g$ is $R - \{0\}$.**

Ans. False : We have,

$$f(x) = 1/x$$
$$g(x) = \tan x$$
$$f + g = \frac{1}{x} + \tan x$$
$$= \frac{1 + x \tan x}{x}$$

Its domain is $R - \{0\} = \{x \neq 0\}$.

***(b)* The set**

$$S = \left\{ x + \frac{1}{x} | 0 < x < 1 \right\}$$

is bounded above.

Ans. False.

We have $s = \{ x + \frac{1}{x} | 0 < x < 1$

$a\,sx = 0, s = \infty$ and $x = 1, s = 2$

The set between 2 to ∞. Hence, it is bounded below but not above.

***(c)* The function**

$$f(x, y) = \tan\left(\frac{x^4 - 2y^4}{x^2 + y^2}\right)$$

is a homogeneous function of degree 2.

Ans. True.

$$f(x, y) = \tan\left(\frac{x^4 - 2y^4}{x^2 + y^2}\right)$$

Standard form

$$\frac{dy}{dx} = \tan\left(\frac{x^4 - 2y^4}{x^2 + y^2}\right)$$

Quotiant of homogeneous functions that have the same degree.

***(d)* The function $f(x, y) = (2x + y^3, 3xy^2 + 8)$ is a conservative function.**

Ans. $f(x, y) = (2x + y^3, 3xy^2 + 8)$

Let $M = 2x + y^3$

Let $N = 3xy^2 + 8$

$$\Rightarrow \frac{\partial N}{\partial y} = 3y^2$$

$$\frac{\partial N}{\partial x} = 3y^2$$

$\therefore f(x, y)$ is conservative. **True.**

***(e)* If $u(x, y) = x \sin y$ and $v(x, y) = x \cos y$, then u and v are functionally dependent on the domain D = $\{(x, y) | x > 0|$.**

Ans. True

If $u(x, y) = x \sin y$

and $v(x, y) = x \cos y$

$$u/v\,(x, y) = \frac{x \sin y}{x \cos y}$$
$$= \tan y$$

$\therefore$ So both are functionally dependent.

Q. 2. *(a)* Evaluate:

(i) $\lim_{x \to 0} \frac{3 \sin x - x}{2x^3}$

Ans. $\lim_{x \to 0} \frac{3 \sin x - x}{2x^3}$

$$\frac{\lim_{x\to 0} (3 \sin x - x)}{\lim_{x\to 0} 2x^3} \Rightarrow \frac{\lim_{x\to 0} 3 \sin x - \lim_{x\to 0} x}{\lim_{x\to 0} 2x^3}$$

$$\Rightarrow \quad \frac{3 \lim\limits_{x\to 0} \sin x - \lim\limits_{x\to 0} x}{2 \lim\limits_{x\to 0} x^3}$$

$$\Rightarrow \quad \frac{3 \sin 0 - 0}{2 \times (0)^3}$$

$$\Rightarrow \quad \frac{3 \times 0 - 0}{2 \times 0}$$

$$\Rightarrow \quad \frac{0 - 0}{0}$$

$$\Rightarrow \quad \frac{0}{0}$$

Since $\frac{0}{0}$ is of indeterminate form, apply L′ Hospital's Rule.

(ii) $\lim\limits_{x \to 0^+} (\sin x)^{\sin x}$

Ans. $\lim\limits_{x \to 0^+} (\sin x)^{\sin x}$

$$\text{First let } y = (\sin x\ \lambda)^{\sin x}$$

$$\text{Then } ln\,(y) = \sin x\ ln\,(\sin x)$$

$$= \frac{ln\,(\sin x)}{cosec\ (x)}$$

Now use L′ Hospital's Rule to evaluate the limit of this expression (it is an $\frac{\infty}{\infty}$ indeterminate form).

$$\lim_{x \to 0^+} \frac{ln\,(\sin x)}{\text{cosec}\,(x)} = \lim_{x \to 0^+} \frac{\frac{1}{\sin x}.\cos x}{-\cot x.\ \text{cosec}\ x}$$

$$= \lim_{x \to 0^+} (-\tan x)$$

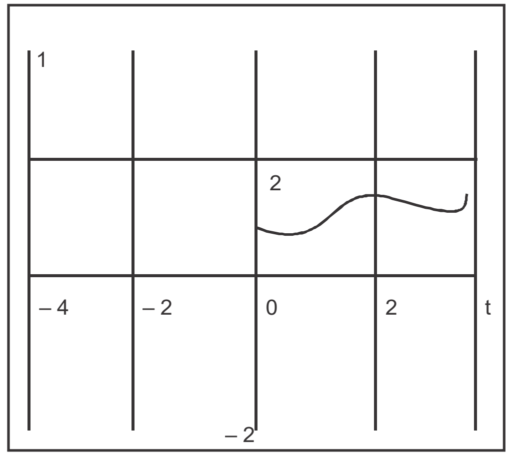
Therefore, $ln \left(\lim\limits_{x \to 0^+} y\right) = \left(\lim\limits_{x \to 0}\right) ln\,(y)$

$$= \lim_{x \to 0^+} \left(\sin x\ ln \cos x\right) = 0$$

Now exponential to implies that:

$$\lim_{x \to 0^+} \left(\sin (x)\right)^{\sin (x)} = \lim_{x \to 0^+} y = e^{o} = 1$$

The graph of $(\sin x)^{\sin x}$ confirms this visually.



(b) **Locate and classify the stationary points of the function**

$$\boldsymbol{f(x, y) = x^2 + y^2 - 6xy + 6x + 3y - 4.}$$

Ans. $f(x, y) = x^2 + y^2 - 6xy + 6x + 3y - 4$

$$f'(x, y) = 2x - 6y + 6 = 0 \quad \text{—(1)}$$

$$f'_2(x, y) = 2y - 6x + 3 = 0 \quad \text{—(2)}$$

From the first (1) we get

$$2(x - 3y + 3) = 0$$
$$x - 3y + 3 = 0$$
$$x = 3y - 3$$
$$x = 3(y - 1)$$

From the second (2) equation we get

$$2y - 6x + 2 = 0$$
$$2y = 6x - 3$$
$$2y = 3(2x - 1)$$
$$y = \frac{3}{2}(2x - 1)$$
$$y = \left(3x - \frac{3}{2}\right)$$
$$x = 3\left\{\left(3x - \frac{3}{2}\right) - 1\right\}$$
$$x = \frac{9}{2}\left(2x - 1 - 1\right)$$
$$x = \frac{9}{2}\left(2x - 2\right)$$
$$x = 9\left(x - 1\right)$$
$$x = 9x - 9$$
$$9x - x = 0$$
$$8x = 9$$

Sample Preview
of
The Chapter

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ADVANCED CALCULUS

1 R_∞ AND RN

Infinite Limits

1

INTRODUCTION

In mathematics, the affinely extended real number system is obtained from the real number system R by adding two elements + ∞ and – ∞ [Read as positive infinity and negative infinity]. These new elements are not real numbers. It is useful in describing various limiting behaviours in calculus and mathematical analysis, especially in the theory of measure and integration. The affinely extended real number systems denoted R or [– ∞, + ∞].

In this chapter, we extended real number system by adding two new symbols ∞ and – ∞. We defined $\lim_{x \to a} f(x) = L$, where a and L is any real number. We used the algebra of limits to calculate the limits of some functions and developed some techniques to calculate these limits.

CHAPTER AT A GLANCE

THE EXTENDED REAL NUMBER SYSTEM R_∞

In treating ∞ and – ∞ as numbers, we are extending the real number system. What we have is

$$R \cup \{-\infty, \infty\} = \{-\infty, +\infty\}$$

This is called the extended real number system. It is sometimes denoted R#.

In the extended real number system, we have the usual order $-\infty < \infty$ and for any real number x, then

$$-\infty < x < \infty.$$

Arithmetic Operations in R∞

Many of the operations we do with real number can be extended to the extended real number system, but not all.

(i) $\infty + \infty = \infty \times \infty = (-\infty)(-\infty) = \infty$

(ii) $-\infty - \infty = (-\infty)\infty = \infty(-\infty) = -\infty$

If x is any real number, then

(i) $\infty + x = x + \infty = \infty$

(ii) $-\infty + \infty = x - \infty = -\infty$

(iii) $\frac{x}{\infty} = \frac{x}{-\infty} = 0$

(iv) $\infty \times x = x \times \infty = \begin{cases} \infty & \text{if } x > 0 \\ -\infty & \text{if } x < 0 \end{cases}$

(v) $(-\infty) \times x = x \times (-\infty) \begin{cases} -\infty & \text{if } x > 0 \\ \infty & \text{if } x < 0 \end{cases}$

However, the following are still indeterminate forms. Their behaviour is unpredictable. Finding what they are equal to require more advanced techniques such as L'Hospital's Rule

(i) $-\infty + \infty$ and $\infty - \infty$

(ii) $0 \times \infty$ and $\infty \times 0$

(iii) $\frac{\infty}{\infty}$

Order Relation in R∞

The order relation on R extends to $\overline{R}$ by defining that for any $x \in R$, we have

$$-\infty < x$$
$$x < \infty$$

and that $-\infty < \infty$ for $a \in R$, let us define intervals.

$$(a, \infty) = \{x \in R \,;\, x > a\}$$
$$(-\infty, a) = \{x \in R : x < a\}$$

Bounds in R_∞

Let C be a non-empty subclass of R. A number $w \in R$ is called an upper bound of C if $x \le w$ for all $x \in$ C. We call C bounded above if it has a upper bound. Similarly $w \in R$ is called a lower bound of C if $w \le x$ for all $x \in c$. We call C bounded below if it has a lower boud. We call C bounded if it has both an upper bound and a lower bound.

Suppose now that C is bounded above, we call a real number S the supremum of C and write S = Sup C if it satisfies the following two conditions:

(a) S is an upper bound of C, and

(b) If w is an upper bound of C the $S \le w$.

Now suppose that C is bounded below. We call a real number S the infimum of C and write S = inf C, if it satisfies the following two conditions:

(a) S is a lower bound of C, and

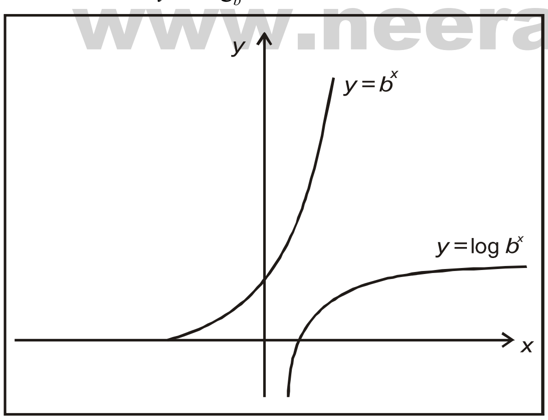
(b) If w is a lower bound of C then $S \ge w$.

Extension of Exponential and Logarithmic Function to R_∞

An exponential function is a function of the form $y = f(x) = b^x$, where b is called the base of the exponential, it will strictly positive not equal to 1 and x the independent variable, is called the exponent.

The logarithmic function base b denoted $\log_b x$ is the inverse of b^x. Therefore, we have the following relation

$$y = \log_b x$$



Now R_∞ is extended to power function b^x.

$$b^\infty = \begin{cases} \infty & b > 1 \\ 0 & ,0 < b < 1 \end{cases}$$

$$b^{-\infty} = \begin{cases} 0 & b > 1 \\ \infty & ,0 < b < 1 \end{cases}$$

INFINITE LIMITS

In Limit Notation $\lim\limits_{x \to \infty}$, the symbol $x \to \infty$ indicates that x increases indefinitely or x approaches ∞. Let f be a function such that $f(x)$ is defined for sufficiently large x suppose that $f(x)$ is arbitrarily large if x is sufficiently large.

Then we write $\lim\limits_{x \to \infty} f(x) = \infty$

because ∞ is not a real number $\lim\limits_{x \to \infty} f(x) = \infty$ does not mean the limit exists.

Instead of $\lim\limits_{x \to \infty} f(x) = \infty$, we also write $f(x) \to \infty$ as $x \to \infty$.

Infinite Limits as the Independent Variable $x \to a \in R$

Definition: Let $f(x)$ be a real valued function defined is an open interval $]a - h, a + h[$ except possibly at a, then $f(x)$ is said to positive real number $\delta < h$, such that

$$0 < |x - a| < \delta \Rightarrow f(x) > M$$

Definition: Let $f(x)$ be a real valued function defined is an open interval $]a - h, a + h[$ except possibly at a. Then $f(x)$ is said to the limit $-\infty$ as x approaches a, if given any real number m, there exists a positive real number $\delta < h$ such that

$$|0| < |x - a| < \delta \Rightarrow f(x) < m.$$

One Sided Infinite Limits

We consider limits at a point on the real line by letting x approach a. Because x can approach a from the left sides or from the right side. We have left side and right side limits. They are called one sided limits.

Let $a \in R$ and let f be a function such that $f(x)$ is defined for x sufficiently close to and greater than a. Supper L is a real number satisfying.

If $f(x)$ is arbitrarily close to L if x is sufficiently close to and greater than a.

Then we say that L is the right side limit of f at a we write $\lim\limits_{x \to a^+} f(x) = L$

If a function f is defined on the left side of a, we can consider its left side limit. The notation $\lim\limits_{x \to a^-} f(x) = L$ means that $f(x)$ is arbitrarily close to L if x sufficiently close to and less than a.

Similar to $\lim\limits_{x \to \infty} f(x) = \infty$. We have the following notations:

(i) $\lim\limits_{x \to a^+} f(x) = \infty$

(ii) $\lim_{x\to a^+} f(x) = -\infty$

(iii) $\lim_{x\to a^-} f(x) = \infty$

(v) $\lim_{x\to a^-} f(x) = -\infty$

Example 1. Show that $\lim_{x\to 3^+} \frac{1}{\sqrt{x^2-9}} = \infty$

Sol. Clearly $\sqrt{x^2-9} > 0$ for $x > 3$ and

$\sqrt{x^2-9} \to 0$ as $x \to 3^+$

therefore $\lim_{x\to 3^+} \frac{1}{\sqrt{x^2-9}} = \infty$

Example 2. Show that $\lim_{x\to 2} 4x = 8$

Sol. For $|4x - 8| < e$ if $|4(x-2)| < e$
i.e. if $4|x-2| < e$ i.e. if $|x-2| < e/4$
Thus $\delta = e/4$, therefore every $e > 0$, a number $\delta > 0$ where $\delta = e/4$ satisfying $|4x-8| < e$ for all $|x-2| < d$

Hence $\lim_{x\to 2} 4x = 8$.

Example 3. Find the right hand and the left hand limit of function f as follows

$$f(x) = \begin{cases} \frac{(x-4)}{x-4} & , x \neq 4 \\ 0 & , x = 4 \end{cases}$$

Sol. When $x > 4$ $|x-4| = x-4$

$$\therefore \lim_{x\to 4^+} f(x) = \lim_{x\to 4^+} \frac{|x-4|}{x-4}$$

$$= \lim_{x\to 4^+} \frac{x-4}{x-4}.$$

$= 1$ (Right hand limit)

Again when $x < 4$ $|x-4| = -(x-4)$

$$\therefore \lim_{x\to 4^-} f(x) = \lim_{x\to 4^-} \frac{-(x-4)}{(x-4)}$$

$= -1$ (Left hand limit)

$$\therefore \lim_{x\to 4^-} f(x) \neq \lim_{x\to 4^+} f(x)$$

$\therefore \lim_{x\to 4} f(x)$ DNE.

Limit as the Independent Variable Tends to ∞ or $-\infty$.

Definition: Let f be a real-valued function defined for all $x > r$, where r is some real number.

(i) *f(x)* is said to approach a real number L as x approaches ∞, i.e., $\lim_{x\to\infty} f(x) = L$, if given any real number $\varepsilon > 0$ there exists a real number G (depending on ε), $G > r$, such that
$x > G \to |f(x) - L| < \varepsilon$.

(ii) *f(x)* is said to approach ∞ as x approaches ∞, i.e., $\lim_{x\to\infty} f(x) = \infty$, if given any real number M there exists a real number G (depending on M), $G > r$ such that
$x > G \Rightarrow |f(x)| > M$.

(iii) The function *f(x)* is said to approach $-\infty$ as x approaches ∞, i.e., $\lim_{x\to\infty} f(x) = -\infty$, if given any real number m there exists a real number G (depending upon m), $G > r$, such that
$x > G \Rightarrow f(x) < m$.

Definition : Let *f(x)* be a real-valued function defined for all $x < r$, where r is some real number.

(i) The function *f(x)* is said to approach a real number L as x approaches ∞, if given any real number $\varepsilon > 0$ there exists a real number g (depending on ε), $g < r$, such that
$x < g \Rightarrow |f(x) - L| < \varepsilon$.

(ii) The function *f(x)* is said to approaches ∞ as x approaches $-\infty$ if given any real number g there exists a real number g (depending on M), $g < r$, such that
$x < g \Rightarrow f(x) > M$.

(iii) The function *f(x)* is said to approach ∞ as x approaches $-\infty$, if given any real number m there exists a real number g (depending on m), $g < r$, such that
$x < g \Rightarrow f(x) < m$.

Example 4. Prove that $\lim_{x\to 0} \frac{-1}{\sin^2 x} = -\infty$

Sol. Let $M > 0$ be given, if $0 < \delta < \pi/2$, then for $x \in]-\delta, \delta[$, $x \neq 0$ we have $\sin^2 x < x^2$

$$\frac{-1}{\sin^2 x} < \frac{-1}{x^2} < \frac{-1}{\delta^2}$$

Thus if $0 < \delta < \min\left\{\frac{\pi}{2}, \frac{1}{\sqrt{M}}\right\}$ then

$$0 < |x| < \delta \Rightarrow \frac{-1}{\sin^2 x} < \frac{-1}{\delta^2} < -M$$

$$\lim_{x\to 0} \frac{-1}{\sin^2 x} = -\infty$$

Example 5. $\lim\limits_{x\to\infty} \dfrac{3x^2}{x^2+2} = 3.$

Sol. Let $\varepsilon > 0$ be given such that $\left|\dfrac{3x^2}{x^2+2} - 3\right| < \varepsilon$

$$\Rightarrow \left|\frac{3x^2 - 3x^2 - 6}{x^2+2}\right| < \varepsilon$$

$$\Rightarrow \frac{6}{x^2+2} < \varepsilon$$

$$\Rightarrow \frac{6}{x^2} < \varepsilon$$

$$\Rightarrow \frac{1}{x^2} < \frac{\varepsilon}{6}$$

$$\Rightarrow x^2 > \frac{6}{\varepsilon} \Rightarrow x > \sqrt{\frac{6}{\varepsilon}}$$

Choose $\sqrt{\dfrac{6}{\varepsilon}} = \text{M}$

Hence $\left|\dfrac{3x^2}{x^2+2} - 3\right| < \varepsilon$ whenever $x > \text{M}$

So $\lim\limits_{x\to\infty} \dfrac{3x^2}{x^2+2} = 3$

Algebra of Limits

We state below some of the important theorems without proof. They will be found very useful in finding limits.

(i) If $\operatorname*{Lim}\limits_{x\to a} f(x) = \mu$ and $\operatorname*{Lim}\limits_{x\to a} f(x) = l_2$ then $l_1 = l_2$

i.e. limit is unique

(ii) $\operatorname*{Lim}\limits_{x\to a} f(x) = l$ if and only if $\operatorname*{Lim}\limits_{x\to a}[f(x) - l] = 0$

(iii) If $\operatorname*{Lim}\limits_{x\to a} f(x) = l$ then $\operatorname*{Lim}\limits_{x\to a} k\, f(x) = kl$

where k is constant

(iv) If $\operatorname*{Lim}\limits_{x\to a} f(x) = 0$ and $\operatorname*{Lim}\limits_{x\to a} g(x) = k$ then

$\operatorname*{Lim}\limits_{x\to a} [f(x) \cdot g(x)] = 0$

(v) $\operatorname*{Lim}\limits_{x\to a} f[g(x)] = f[\lim\limits_{x\to a} g(x)]$

(vi) If $f(x) \le g(x) \le h(x)$ and

$\operatorname*{Lim}\limits_{x\to a} f(x) = \lim\limits_{x\to a} h(x) = l$ then $\lim\limits_{x\to a} g(x) = l$.

Example 6. $\lim\limits_{x\to\infty} \dfrac{1}{\log x} = 0$

Sol. Let $\varepsilon > 0$ be given such that $\left|\dfrac{1}{\log x} = 0\right| < \varepsilon$

$\dfrac{1}{\log x} < \varepsilon \Rightarrow x > e^{\varepsilon}$

Thus if $\text{M} = e^{\varepsilon}$

Then $x > \text{M} \Leftrightarrow \left|\dfrac{1}{\log x} - 0\right| < \varepsilon$

Hence $\lim\limits_{x\to\infty} \dfrac{1}{\log x} = 0$

Example 7. $\lim\limits_{x\to\infty} \dfrac{e^x - 1}{e^x + 1} = 1$

Sol. Dividing Numerator and Denominator by e^x, we get

$$\lim_{x\to\infty} \frac{\dfrac{e^x}{e^x} - \dfrac{1}{e^x}}{\dfrac{e^x}{e^x} + \dfrac{1}{e^x}} \Rightarrow \lim_{x\to\infty} \frac{1 - e^x}{1 + e^{-x}}$$

$$\Rightarrow \frac{1-0}{1+0}$$

$$\Rightarrow 1$$

Example 8. $\lim\limits_{x\to\infty} \dfrac{x^5 + 4x^3 + 3x^2 + 7}{2x^5 + x^4 + 3x + 6} = \dfrac{1}{2}$

Sol. Dividing Numerator and Denominator by x^5, we get

$$\lim_{x\to\infty} \frac{1 + \dfrac{4}{x^2} + \dfrac{3}{x^3} + \dfrac{7}{x^5}}{2 + \dfrac{1}{x} + \dfrac{3}{x^4} + \dfrac{6}{x^5}}$$

$$= \lim_{x\to\infty} \frac{\left(1 + \dfrac{4}{x^2} + \dfrac{3}{x^3} + \dfrac{7}{x^5}\right)}{\left(2 + \dfrac{1}{x} + \dfrac{3}{x^4} + \dfrac{6}{x^5}\right)}$$

$$= \frac{\lim\limits_{x\to\infty} 1 + \lim\limits_{x\to\infty} \dfrac{4}{x^2} + \lim\limits_{x\to\infty} \dfrac{3}{x^2} + \lim\limits_{x\to\infty} \dfrac{7}{x^5}}{\lim\limits_{x\to\infty} 2 + \lim\limits_{x\to\infty} \dfrac{1}{x} + \lim\limits_{x\to\infty} \dfrac{3}{x^4} + \lim\limits_{x\to\infty} \dfrac{6}{x^5}}$$