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DIFFERENTIAL EQUATIONS

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M.Sc, M.Phil (Math)

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QUESTION PAPER

(June – 2015)

(Solved)

DIFFERENTIAL EQUATIONS

Time: 2 hours] *[Maximum Marks : 50*
(Weightage 70%)

Note: *Question No. 1 is **compulsory.** Answer any **four** questions from the remaining Question No. 2 to 7. Use of calculators is **not** allowed.*

Q. 1. State whether the following statements are *True* or *False.* Justify your answer with the help of a short proof or a counter-example.

(a) **The differential equation**

$$x\,dx + y\,dy = \frac{a^2(x\,dy - y\,dx)}{(x^2+y^2)}$$

is exact .

Ans. False

$$x\,dx + y\,dy = \frac{a^2(x\,dy - x\,dx)}{(x^2+y^2)}$$

$$\Rightarrow x(x^2+y^2)\,dx + y(x^2+y^2)\,dy = a^2x\,dy - x\,dx$$

$$\Rightarrow [x^3+xy^2+x]dx = [a^2x - x^2y - y^3]\,dy$$

$$M = x^3+xy^2+x$$

$$N = x^2y+y^3-a^2x$$

$$\frac{\partial M}{\partial y} = 2xy$$

$$\frac{\partial N}{\partial x} = 2xy - a^2$$

Hence, $= \frac{\partial M}{\partial d} \neq \frac{\partial N}{\partial x}$

This equation is not exact.

(b) **The fundamental solutions for the differential equation.**

$$x^2\frac{d^2y}{dx^2} - x\frac{dy}{dx} + y = 0$$

are *x* and *x ln x*.

Ans. True,

$$x^2\frac{d^2y}{dx^2} - x\frac{dy}{dx} + x = 0$$

$$(x^2D^2 - xD + 1)\,y = 0$$

Let $x = e^z$

$$(D_1(D_1-1) - D_1 + 1)\,y = 0$$

$$(D_1{}^2 - 2D_1 + 1)\,y = 0$$

Auxiliary Equation

$$m^2 - 2_m + 1 = 0$$

$$\Rightarrow (m-1)^2 = 0$$

$$\Rightarrow m = 1, 1$$

$$y = (c_1 + c_2 z)\,e^z$$

$$\Rightarrow y = (c_1 + c_2\,lnx)\,x.$$

(c) **The primitive of the partial differential equation $\sqrt{p}+\sqrt{q} = 2x$, is :**

$$\frac{1}{6}(2x-a)^3 + a^2y + b$$

Ans. True,

$$f = \sqrt{P} + \sqrt{q} - 2x$$

$$f_p = \frac{1}{2\sqrt{P}} \quad fq = \frac{1}{2\sqrt{q}}$$

$$f_x = -2$$

$$\frac{dx}{f_p} = \frac{dy}{fq} = \frac{dz}{-(pf_p + qf_q)}$$

$$= \frac{dp}{f_x + pf_z} = \frac{dq}{f_y\,qf_z}$$

$$\frac{dx}{\frac{1}{1\sqrt{P}}} = \frac{dy}{\frac{1}{2\sqrt{q}}} = \frac{dz}{-\left(\frac{\sqrt{P}}{2} + \frac{\sqrt{q}}{2}\right)}$$

$$= \frac{dp}{-2} = \frac{dq}{0}$$

$$\Rightarrow \quad \frac{d_p}{-2} = \frac{dq}{0}$$

$$\Rightarrow \quad q = c$$

$$\sqrt{P} = 2x - a$$

$$P = (2x-a)^2$$

$$dz = pdx + qdy$$

$$\Rightarrow \quad dz = (2x-a)^2 + a^2 \text{ dy}$$

$$\Rightarrow \quad z = \frac{(2x-a)^3}{6} + a^2 y + b.$$

***(d)* For all real values of x, the differential equation $x^2 \frac{\partial^2 u}{\partial x^2} - \frac{\partial^2 u}{\partial_{y^2}} + u = 0,$ is elliptic.**

Ans. False,

$$x^2 \frac{\partial^2 u}{\partial x^2} - \frac{\partial^2 u}{\partial y^2} + u = 0$$

$$s^2 - u\text{RT} = 0 - u \times x^2 \times (-1)$$

$$= ux^2 > 0$$

It is hyperbolic not elliptic.

(e) For the IVP, $\frac{dy}{dx} = f(x, y), y(x_0) = y_0$, the continuity of f(x, y) and $\frac{\partial f}{\partial y}$ guarantees the uniques solution of the problem.

Ans. True, Property of Lipritehz Condition.

Q. 2. Solve the following differential equations.

(a) $\quad \frac{dy}{dx} + xy = y^2 e^{x^2/2} \cos x$

Ans. $\quad \frac{dy}{dx} + xy = y^2 e^{x^2/2} \cos x$

$$\Rightarrow \quad \frac{1}{y^2}\frac{dy}{dx} + \frac{x}{y} = e^{x^2/2} \cos x$$

$$\Rightarrow \quad y^{-2}\frac{dy}{dx} + xy^{-1} = e^{x^2/2} \cos x$$

Let $\quad y^{-1} = t$

$$-\frac{1}{y^2} dy = dt$$

$$\Rightarrow \quad -\frac{dt}{dx} + xt = e^{x^2/2} \cos x$$

$$\Rightarrow \quad \frac{dt}{dx} - xt = e^{x^2/2} \cos x$$

$$\text{I.F.} = e^{-\int x dx} = e^{-x^2/2}$$

Solutions,

$$t.\, e^{-x^2/2} = \int (e^{x^2/2} \cos x)(e^{-x^2/2})\, dx$$

$$\Rightarrow \quad t\, e^{-x^2/2} = \int \cos x\, dx$$

$$\Rightarrow \quad t\, e^{-x^2/2} = \sin x + c$$

$$\Rightarrow \quad t = e^{x^2/2} \sin x + ce^{x^2/2}$$

$$\Rightarrow \quad y^{-1} = e^{x^2/2} \sin x + ce^{x^2/2}$$

$$\Rightarrow \quad y\, e^{x^2/2} \sin x + cye^{x^2/2} = 1.4$$

(b) $\frac{d^2 y}{dx^2} + \frac{1}{x}\frac{dy}{dx} = \frac{12\, ln\, x}{x^2}$

Ans.

$$\frac{d^2 y}{dx^2} + \frac{1}{x}\frac{dy}{dx} = \frac{12\, lnx}{x^2}$$

$$\text{I.F.} = e^{\int \frac{1}{x} dx} = e^{\ln x} = x$$

$$y \times x = \int \frac{12\, lnx}{x^2} \times x\, dx$$

$$\Rightarrow \quad \text{y} \times \text{x} = \int \frac{12\, lnx}{x} dx$$

Let, $\quad lnx = t$

$$\Rightarrow \quad \frac{1}{x} dx = dt$$

$$\Rightarrow \quad xy = \int 12\, t\, dt$$

$$\Rightarrow \quad xy = 6t^2 + c$$

$$\Rightarrow \quad xy = 6\,(lnx)^2 + c$$

Sample Preview of The Chapter

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DIFFERENTIAL EQUATIONS

Ordinary Differential Equations of First Order

1

The Nature of Differential Equations

INTRODUCTION

Many practical problems in science and engineering are formulated by finding how one quantity is related to, or depends upon, one or more quantities defined on the problem. Often, it is easier to model a relation between the rates of changes in the variables rather than between the variables themselves. The study of this relationship gives rise to differential equations. The differential equations are used to model physical systems.

CHAPTER AT A GLANCE

BASIC CONCEPTS

Definition: An equation that involves one or more dependent variables and their derivatives with respect to one or more independent variables is called a differential equation.

For example:

(i) $\frac{dy}{dx} = 6x^2$

(ii) $\frac{d^2y}{dx^2} + 16y = 2x$

(iii) $\frac{\partial z}{\partial x} + xy\frac{\partial z}{\partial y} = zx$

are all differential equations.

NOTE 1. Equations of the type

$$\frac{d}{dx}(xy) = y + x\frac{dy}{dx}$$

are not differential equations. In this equation if you expand the left hand side then you will find the left hand side is the same as the right hand side. Such equations are called **identities.**

NOTE 2: The equation

$$\left(\frac{dy}{dx}\right)_x = (y)_{x+1}$$

is not a differential equation. This is because y is evaluated at $(x+1)$ whereas $\frac{dy}{dx}$ is evaluated at x.

The **classification** of differential equations is based on the nature of the dependent variables and their derivatives in the equations. There are **three** types of differential equations**:**

1. Ordinary Differential Equation (ODE): A differential equation which contains derivatives with respect to a single independent variable.

Examples:

$$\frac{dy}{dx} = 3x^2 + 2x \quad ...(1)$$

$$\frac{d^2y}{dx^2} + 6y = 2x^2 \quad ...(2)$$

$$y = x^2 + \frac{dy}{dx} \quad ...(3)$$

are all ordinary differential equations.

The general representation of the ordinary differential equation is:

$$g\left[x, y, \frac{dy}{dx}, \frac{d^2y}{dx^2}, ..., \frac{d^ny}{dx^n}\right] = 0$$

where g is a real valued function.

2. Partial Differential Equation (PDE): An equation that contains two or more independent variables and partial differential co-efficient with reference to any of them.

Examples:

$$x\frac{\partial z}{\partial x} + \frac{\partial z}{\partial y} - z = 0$$

$$y^2\frac{\partial z}{\partial x} + xy\frac{\partial z}{\partial y} - nzx = 0$$

$$\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = 0$$

3. Total Differential Equation: Those equations that contain two or more dependent variables together with their derivatives with respect to a single independent variable that may or may not exist explicitly in the equation.

Examples:

$$yz\,dx + xz\,dy + xy\,dz = 0$$

$$\frac{x\,dx + y\,dy + z\,dz}{\sqrt{x^2 + y^2 + z^2}} + x\,dx + dy + 3\,dz = 0$$

Order of Differential Equation: The order of a differential equation is the order of the highest derivative appearing in it.

For example, equation

$$\frac{d^2y}{dx^2} + \frac{dy}{dx} = \sin x \quad ...(4)$$

is of second order because the highest derivatives is $\frac{d^2y}{dx^2}$, which is of second order. And the equation $\frac{dy}{dx} = 3x^2 + 6x + 9$ is of first order as the order of the highest derivative is one.

Degree of a Differential Equation: The degree of an equation is the degree of that highest derivative, when the differential coefficients are free from radicals and fractions. For example, the equation $y\frac{dy}{dx} = \sqrt{x}\left(\frac{dy}{dx}\right)^2 + 4$ is of second degree. In the earlier example, degree of the equation (4) is one as the degree of highest derivative is one.

We now classify the differential equations depending upon the degree of dependent variable and its derivatives into two classes, namely linear and non-linear.

Linear and Non-linear Differential Equations: When in a differential equation, either ordinary or partial differential equation, the dependent variable and its derivatives occur in the first degree only we call the equation **linear.** When a differential equation is not linear we call it **non-linear.**

Examples: The equation $\frac{d^2y}{dx^2} + 3\frac{dy}{dx} = 54y$ is an ordinary linear differential equation as the degree of dependent variable and its derivatives is one. However, the equation $x^2\frac{dy}{dx} + y^2 = 1$ is an ordinary non-linear differential equation.

Similarly,

Equation $\frac{\partial z}{\partial x} + \frac{\partial z}{\partial y} = z$ is a linear partial differential equation and the equation $\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} = \frac{\partial^2 z}{\partial x \partial y}$ is a non –linear partial differential equation.

SOLUTION OF A DIFFERENTIAL EQUATION

A function $y = f(x)$ is called solution of a differential equation on an interval I, if $f(x)$ is continuous and differential (required number of times) throughout the interval and if substitution of $y = f(x)$ and its derivatives into the equation reduces to an identity.

For example, $y = ce^{2x}$ is a solution of $\frac{dy}{dx} = 2y$ because by putting $y = ce^{2x}$ and $\frac{dy}{dx} = 2ce^{2x}$, the given differential equation reduces to the identity $2ce^{2x} = 2ce^{2x}$.

Example: Show that equation of the family of curve $y = e^x$ (A cos x + B sin x), where A and B are arbitrary constants, is the solution of the differential equation $\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + 2y = 0$.

Solution: Given that

$$y = e^x(A\cos x + B\sin x) \quad ...(5)$$

Differentiating (5) with respect to 'x', we get

$$\frac{dy}{dx} = e^x(-A\sin x + B\cos x) + e^x(A\cos x + B\sin x)$$

i.e. $\frac{dy}{dx} = e^x(-A\sin x + B\cos x) + y$, Using (5) ...(6)

Differentiating (6) again with respect to 'x', we get

$$\frac{d^2y}{dx^2} = -e^x(\mathrm{A}\cos x + \mathrm{B}\sin x)$$

$$+ e^x(-\mathrm{A}\sin x + \mathrm{B}\cos x) + \frac{dy}{dx} \quad ...(7)$$

Now from (6),

$$e^x(-\mathrm{A}\sin x + \mathrm{B}\sin x) = \frac{dy}{dx} - y \quad ...(8)$$

Hence eliminating A and B from (5), (7) and (8), we get

$$\frac{d^2y}{dx^2} = -y + \frac{dy}{dx} - y + \frac{dy}{dx} \text{ or } \frac{d^2y}{dx^2} - 2\frac{dy}{dx} + 2y = 0$$

which shows that y is the solution of the given equation.

To obtain the solution of a differential equation, we integrate it as many times as the order of the differential equation, since each integration reduces the order of the differential equation by one. Also, each integration, introduces one arbitrary constant in the solution. Accordingly, we classify various types of solutions of an ordinary differential equation as follows:

General Solution: The solution of the nth order differential equation that contains n arbitrary constants is called its general solution. General solution is also called complete integral or complete primitive of the differential equation.

Particular Solution: Any solution obtained from the general solution, by giving particular values to the arbitrary constants, is called a particular solution.

For example, $y = \mathrm{A}e^{2x} + \mathrm{B}e^{-2x}$, involving two arbitrary constants A and B, is the general solution of the differential equation $\frac{d^2y}{dx^2}$ = 4y whereas $y = e^{2x} + e^{-2x}$ is its particular solution (taking A = 1 and B = 1).

Singular Solution: In the case of most of the differential equations, every solution can be obtained from the general solution by assigning suitable values to the arbitrary constants. However, in some cases there exists a solution which cannot be obtained from the general solution. Such a solution is called a singular solution.

For example, it can be verified that $y = cx + c^2$ is the general solution of the differential equation

$$\left(\frac{dy}{dx}\right)^2 + x\frac{dy}{dx} = y \quad ...(9)$$

the general solution is a parameter family of straight lines, one straight line for each value of c. However, we find that (by substitution) $4y + x^2 = 0$ is also a solution of the differential equation, which cannot be obtained from the general solution.

REMARK: Not all differential equations that we come across have unique solution or a family of solutions. For example, the differential equation

$$\left|\frac{dy}{dx}\right| + |y| = 0 \quad ...(10)$$

has only the trivial solution, that is $y = 0$.

The differential equation

$$\left|\frac{dy}{dx}\right| + |y| + c = 0 \qquad c > 0 \quad ...(11)$$

has no solution.

From the above discussion we now try to find the conditions under which the solution of a given differential equation exists and is unique. For that consider the following theorem:

Theorem 1: (Existence-Uniqueness)

Let $f(x, y)$ be continuous in a domain D of the (x, y) plane and let M be a constant such that

$|f(x, y)| \leq$ M in D. Let $f(x, y)$ satisfy in D the Lipschitz condition in y namely ...(12)

$$|f(x, y_1) - f(x, y_2)| \leq \mathrm{K}\,|y_1 - y_2| \quad ...(13)$$

where the constant K is independent of x, y_1, y_2:

Let the rectangle R, defined by

$$|x - x_0| \le h, \ |y - y_0| \le k \qquad \ldots(14)$$

lie in D, where Mh < k. Then, for $|x - x_0| \le h$, the differential equation $\frac{dy}{dx} = f(x, y)$ has a unique solution $y = y(x)$ for which $y(x_0) = y_0$.

The conditions stated in Theorem 1 are sufficient but not necessary and can be relaxed.

Example: Investigate the existence of solution of the $\frac{dy}{dx} = 2x^2 + 3y^2, y(0)$ over the rectangle $|x| \le 1, |y-1| \le 1$.

Solution: here $f(x, y) = 2x^2 + 3y^2$, $x_0 = 0$ and $y_0 = 1$. The given rectangle is $|x - 0| \le 1 and |y-1| \le$ or $-1 \le x \le 1$ and $0 \le y \le 2$. Now $f(x, y)$ is continuous everywhere in the rectangle. Further

$$|f(x, y)| = |2x^2 + 3y^2| \le 14, \quad -1 \le x \le 1, \ 0 \le y \le 2$$

Therefore, at least one solution of the IVP exists. Now, M = 14 and

$h = \min\{a, b/M\} = \min\{1, 1/14\} = 1/14$.

Solution exists for all x at least in the interval {-1/14, 1/14}.

FAMILY OF CURVES AND DIFFERENTIAL EQUATIONS

Let y and x be the dependent and the independent variables respectively. The equation

$$f(x, y, c) = 0 \qquad \ldots(15)$$

containing one arbitrary constant c, represents a family of curves. For example, the equation $x^2 + y^2 = r^2$ where r is arbitrary represents a circle with centre at the origin and radius r. The equation

$$g(x, y, c, d) = 0 \qquad \ldots(16)$$

containing two arbitrary constants c and d also represents a family of curves. For example, the equation $y = mx + k$, where m and k are arbitrary constants represents a two-parameter family of straight lines having slope m and passing through the point $(0, k)$.

To eliminate the arbitrary constant c in (15), we need two equations. One equation is given by Eq. (16) itself and the second equation is obtained by differentiating Eq. (15) with respect to x. On eliminating c from two equations, we obtain an equation containing x and y which is a first order differential equation.

DIFFERENTIAL EQUATIONS ARISING FROM PHYSICAL SITUATIONS

As we know that there are many problems of physical and engineering interest that give rise to differential equations. For example

Example: A porous pot containing a solution of a substance of concentration of x mgcm-3 is placed in a large vessel containing the same solution but of higher concentration c mgcm-3. The concentration of the solution in the pot will increase due to diffusion. Assuming that c is constant, the rate of increase of concentration of the solution in the pot is proportional to the difference in concentration. Thus it satisfies the differential equation $\frac{dx}{dt} = k(c - x)$ where k is a positive constant.

SOLVED EXERCISES

Ex.1. Which of the following are differential equations? Which of the differential equations are ordinary and which are partial?

(*a*) $\left(\frac{d^2y}{dx^2}\right)^3 + x\frac{dy}{dx} + y^3 = 5x + 2$

(*b*) $\frac{dy}{dx} = \int_0^x \sin[xy(s)]ds$

(*c*) $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = 0$

(*d*) $\frac{dy(x)}{dx} = 5x.y(x+1)$

(*e*) $\frac{d^2y}{dx^2} + \frac{dy}{dx} + y = \int_0^1 \sin[xy(s)]ds$

(*f*) $\frac{\partial^2 w}{\partial t^2} = a^2 \frac{\partial^2 w}{\partial x^2}$

Sol.

(*a*) It is ODE. *(By definition)*

(*b*) It cannot be a differential equation as the right hand side of the Eq. has an unknown function y appearing inside an integral. Also in this case the value of y on the right hand side of the Eq. depends on the interval 0 to x, whereas in a differential equation, the unknown y has to be evaluated only at x.