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REAL ANALYSIS

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REAL ANALYSIS

Sample Preview of the Solved Sample Question Papers

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QUESTION PAPER

(June-2016)

(Solved)

REAL ANALYSIS

Time: 2 hours] *[Maximum Marks: 50*
(Weightage: 70%)

Note: *Attempt **five** questions in all. Question No. **1** is **compulsory**. Answer any **four** questions out of questions no. 2 to 7. Use of calculator is not allowed.*

Q. 1. Are the following statements *True* or *False*? Give reasons for your answers.

(*a*) 0 is the supremum of the set {–*n* : *n* is a natural number}.

Ans. *True:* 0 is the supremum of the set {–*n* : *n* is a natural no.}, because the negative real no. do not have a greatest element and their supremum is 0.

(*b*) A necessary condition for a function *f* to be integrable is that it is continuous.

Ans. *True*: F is Riemann integrable if $\varepsilon > 0$.

We set

$$\varepsilon_0 = 0 \ \frac{\varepsilon}{(b-a)}$$

Since f is continuous on $[a, b]$, f is uniformly continuous. Hence there is a $z > 0$.

Such that

$|f(y) - f(x)| < \varepsilon_0$ if $|y - x| < z$

Suppose that $|P| < z$, then it allow that

$$|M_i - m_i| \leq \varepsilon_0 \ (1 \in i \in n)$$

Hence U (P, f) – L. (P, f)

$$= \sum_{i=1}^{n} (M_i - m_i)\Delta x_i \leq \varepsilon_0 (b - 0)$$

$$= \varepsilon$$

(*c*) The greatest integer function [*x*] defined on R is derivable in the interval $\left[\frac{1}{2}, \frac{3}{4}\right]$.

Ans. *True:* $F(x) = [x] \to R$ is derivable in the interval $\left[\frac{1}{2}, \frac{3}{4}\right]$.

(*d*) $\lim\limits_{x \to \frac{\pi}{2}} \dfrac{\cot x}{x - \frac{\pi}{2}}$ does not exist.

Ans. *True:* $\lim\limits_{x \to \frac{\pi}{2}} \dfrac{\cot x}{x - \frac{\pi}{2}}$ does not exist.

$S = R - n\pi$

$n \in Z \, n \geq 0$, when

$\sin x \neq 0$.

(*e*) The series $\sum\limits_{n=1}^{\infty} \left(\frac{e}{\pi}\right)^n$ is convergent.

Ans. *True:* Let $u_n = \left(\frac{e}{\pi}\right)^n$

$$\lim_{x \to \infty} (u_n) = \lim_{x \to \infty} \left(\frac{e}{\pi}\right)$$

$$0 < 1$$

$$\sum u_n = \sum \left(\frac{e}{\pi}\right)^n \text{ converges.}$$

Q. 2. (*a*) Using the Principle of Mathematical Induction, show that

$1 + 3 + 5 + 7 + ... + (2n - 1) = n^2, \forall\, n \in$ N.

Sol. Step (*i*): For $n = 1$ equation is true. Since $1 = 1^{2.}$

Step (*ii*): Suppose equation is true. For some $n = k \geq$ that is $1 + 3 + 5 + ... + (2\,k - 1) = k^2$

Step (*iii*): Prove that equation is true. For $n = k + 1$ that is

$1 + 3 + 5 + ... + (2k - 1) + (2k + 1) = (k + 1)^2$

We have

$\Rightarrow\ 1 + 3 + 5 + ... + (2k - 1) + (2k + 1)$

$= k^2 + (2k + 1)$ ... Using Step (*ii*)

$= (k + 1)^2$

So that equation is

$1 + 3 + 5 + 7 + ... + (2n - 1) + n^2\ \forall\ n \in$ N

(*b*) Show that the function f: [0, 1] $\rightarrow$ R defined by

$$f(x) = \begin{cases} 1 & \text{when } x \text{ is rational} \\ 2 & \text{when } x \text{ is irrational} \end{cases}$$

is not Riemann integrable.

Ans. Let 2 is a rational number there $f(2) = 0$. Since between any two rational numbers there are infinite numbers of rational as well as rational numbers We have irrational no. x as close 2 as we please. There are points x in every neighbourhood of 2 such that

$|f(x) - f(2)| = 0$

Since all irrational numbers $f(x) = 1$ for any $\varepsilon = 0$, we cannot a no. 2 . $b > 0$ such that

$|f(x) - f(2)| < \varepsilon$ if $|(x - 2| < d$

So that f: [0, 1] $\rightarrow$ R is not Riemann integrable.

(*c*) Show that the function

$$f(x) = |x - 5| + x^2 + 3x + 10$$

is continuous, but is not differentiable at the point $x = 5$.

Sol. $f(x) = x - 5 + x^2 + 3x + 10$

$= x^2 + 4x + 5$

$$f(5) = \lim_{x \to 5}(x^2 + 4x + 5)$$

$= 25 + 20 + 5$

$= 50$

$\therefore f$ is continuous at $x = 5$

$$f'(5) = \lim_{x \to 5}\left[\frac{f(x) - f(5)}{x - 5}\right]$$

$$= \lim_{x \to 5}\left[\frac{x^2 + 4x + 5 - 50}{x - 5}\right]$$

$= 0$

$\therefore f'(s)$ doesn't exist.

$\Rightarrow f'(x)$ is not derivable at $x = 5$.

Q. 3. (*a*) Show that the sequence (f_n) where

$$f_n(x) = \frac{x}{1 + 2nx^2},\ x \in [1, \infty[\quad \text{is uniformly}$$

convergent in [1, ∞[.

Sol. $f_n(x) = \dfrac{x}{1 + 2nx^2},\ x \in [1, \infty[$

We have

$$f(x) = \lim_{x \to \infty} f_n(x)$$

$$= \lim_{x \to \infty} \frac{x}{1 + 2nx^2} = \infty$$

And if $x = 1$, we have

$$f(1) = \lim f_n(1)$$

So $f(x) = \begin{cases} 1 & 1 < x \leq \infty \\ \infty & \infty \end{cases}$

We know that $f_n(x)$ is continuous for all n, so $x \in [1, \infty[$ and $f(x)$ is continuous. So that $f_n(x)$ is uniformly convergent in $[1, \infty[$.

(*b*) Check whether the following sequence (s_n) are Cauchy, where

(*i*) $s_n = 1 + 2 + 3 + ... + n$

Sol. $s_n = 1 + 2 + 3 + ... + n$

$\lim_{n \to \infty}(n) = 0$

So that (s_n) is Cauchy sequence.

(*ii*) $s_n = \dfrac{4n^3 + 3n}{3n^3 + n^2}$

Sol. $s_n = \dfrac{4n^3 + 3n}{3n^3 + n^2}$

Taking limit, we get

Sample Preview
of
The Chapter

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REAL ANALYSIS

REAL NUMBERS AND FUNCTIONS

Sets and Numbers

1

INTRODUCTION

Real analysis studies concepts related into real numbers. So we begin our study with a discussion of the set of real numbers. For this we need to learn about sets and about sets of, numbers.

CHAPTER AT A GLANCE

SETS AND FUNCTIONS

Sets and functions are used to represent the most modern mathematics nowadays. Some important definitions are required to study the set theory.These are discussed below:

Sets

A set is a well-defined collection of objects seen as a single entity.

The following are some examples of sets:

1. Set of natural numbers $\{1,2,3,4,5,6\ldots\}$
2. A collection of playing cards
3. A set of whole numbers $\{0,1,2,3,4\ldots\}$

A set consists of elements for example 0,1,2,3 are the elements of set of whole numbers in the above given example.

A set may be represented in **Tabular form** or **Explicit form** or in **Set builder form** or **Implicit form**.

A set, which contains finite number of elements, is called a FINITE SET.

Set which contains an infinite number of elements is called an INFINITE SET.

If every element of a set A is an element of the set B then we say A IS A SUBSET OF B.

WE WRITE $A \subset B$ OR $B \supset A$.

Every set is a subset of itself.

Definition 1: Equality of Sets

Two sets whose elements are identical is called **Equal Sets**. When two sets A and B are unequal we write $A \neq B$.

If X and Y are two sets then $X = Y$ implies $X \subset Y$ and $Y \subset X$.

If $A \subset B$ but $B \not\subset A$ then we say A is a **Proper Subset** of B.

We write $A \subseteq B$.

A set with no elements is called a NULL SET. It is denoted by the symbol ϕ

Null set is a subset of every set.

If every other set under discussion is a subset of a given set then this set is called the UNIVERSAL SET.

Definition 2: Complement of the Set

If X is the universal set and A is any set, the complement of the set A is defined as the set of all elements in X which are not in A. We write $A^c = \{x/x \in X, x \notin A\}$.

POWER SET of a set A denoted as P(A) is defined as the set of all subsets of A.

Power set of any set contains the null set and the set itself. If A contains n elements the P(A) contains 2^n elements.

Definition 3: Union of Sets

Union of two sets A and B, denoted as $A \cup B$, is the collection of all the elements of A as well as B.

The set $A \cup B$ has all the elements of A as well as B. e.g.

$$A = \{1,2\},\ B = \{4, 5\}$$
$$A \cup B = \{1, 2, 4, 5\}$$

Definition 4: Intersection of Sets

Intersection of two sets A and B, written as $A \cap B$ and is defined as the set of all elements common to both A as well as B.

Example 1: If $A=\{1,2,3\}, B=\{2,3,4,5,6\}$ then $A \cap B=\{2,3\}$

If $A \cap B = \phi$, then sets A and B are called Disjoint or mutually exclusive.

Definition 5: Difference of Two Sets

If A and B are two sets then A – B is defined as those elements of A which are not in B.

Similarly B – A is a set of those elements of B, which are not in A.

Functions

Definition 6: Function

If A and B are two sets then we define $f : A \to B$ as a rule by which, to each elements to each of A, is assigned a unique element of B.

If $f: A \to B$ is a function then A is called DOMAIN and B is called CO-DOMAIN of f.

RANGE OF A FUNCTION

If $f: A \to B$ is a function then it maps ever elements of A to a unique element of B. If $x \in X$ is mapped to $y \in B$ then we write $f(x) = y$.

Range of $f = \{f(x): x \in A\}$

If $f : A \to B$ is a function and range of $f = B$ we say f is a ONTO or SURGECTIVE function.

ONE-TO-ONE OR INJECTIVE

If $f: A \to B$ is a function then we say f is one-to-one or injective if $f(a) = f(b) \Rightarrow a = b$ i.e., two different elements of A have two different image.

BIJECTIVE FUNCTION

Any function, which is surjective and injective, it is called a bijective function.

Definition 7: Identity Function

If f from $A \to B$ under which image of each element is equal to the elements itself it is called an identity function. i.e., $f(x) = x$ for every x in a.

Definition 8: Constant Function

If $f : A \to B$ is a function and every element of A is mapped to a single element of B then f called a constant function.

Definition 9: Equality of Functions

If $f : A \to B$ and $g : A \to B$ are two functions and if for every $x \in A$, $f(x) = g(x)$ then f and g are set to be equal functions.

Definition 10: Composite of Functions

If $f: A \to B$ and $g : B \to C$ are two function then $h: A \to C$, defined as $h(x) = (gof)(x) = g(f(x))$ for every $x \in A$, then this function h is called the composite function of f and g.

Definition 11: Inverse of a Function

A function $f : A \to B$ is said to invertible if there exists a function $g: B \to A$ such that

$(gof)(b) = g(f(b))$ for every B in B and $(fog)(b) = f(g(b))$ for every B in A. Then g is called the inverse of A.

All functions are not invertible. A function is invertible if it is both one-to-one and onto.

Also if a function is both one to one and onto then the function is invertible.

SYSTEM OF REAL NUMBERS

Mathematical analysis studies concepts related in some way to real numbers, so we begin our study of real analysis with a discussion of the real-numbers system. Several methods are used to introduce real numbers. One method starts with the positive integers 1,2,3…. as undefined concepts and uses them to build a larger system, the positive rational numbers (quotient and positive integers) their negatives and the number zero. The rational numbers in turn are used to construct irrational numbers. The rational and irrational numbers together make the real number system. We shall begin our discussion with the natural numbers.

Natural Numbers

Generally the numbers 1,2,3 which are used for counting purpose are called natural numbers. This is the most ancient form of numbers known.

This set begin from 1 and every numbers in it has a next number.

Set of natural numbers is denoted by N

N={1,2,3,4,5…}

This is an infinite set closed with respect to addition and multiplication.

Integers

Set of natural numbers is closed with respect to addition and multiplication. This means when we add or multiply two natural numbers we get another natural number. But if we subtract two natural numbers we need not always get a natural number. Therefore N is not closed w.r.t to subtraction so there was a need to extend N to a set which would be closed w.r.t of subtraction.

$\therefore$ The set z, of interers came into existence, which includes all natural numbers, zero, and negative of all natural numbers. This set z is closed w.r.t addition subtraction and multiplication.

Rational Numbers

The set z of integers is closed with respect to addition, subtraction and multiplication. But it is not closed with respect to division. If we divide an integer by another integer we do not always get an integer. Hence the larger set, which was closed with respect to division also, was developed. This is called the set of rational numbers which includes all, positive and negative integers and all the fractions. We denote the set of rational numbers by Q, which is a set of all the numbers of the form p/q where p and q are both integers and $q \neq 0$.

This set Q seems to be including all types of numbers. But this too is not a complete set as numbers of the form x where $x^2 = 7$ is not a rational number.

Since if $\sqrt{7}$ was a rational number then we can find integer p and q such that

$\sqrt{7} = p/q$ where $q \neq 0$ and p and q has no common factors of squaring this, we have

$$7 = \frac{p^2}{q^2}$$

$$\Rightarrow \quad p^2 = 7q^2 \qquad ...(1)$$

$\Rightarrow p$ is a multiple of 7

$\therefore$ Let $p = 7$ where m is a integers subtraction this in (1)

$$(7m)^2 = 7q^2$$

$$\Rightarrow \quad 7m^2 = q^2$$

$\Rightarrow q$ is a multiple of 7

but according to our assumption p and q have no common factors.

Thus, the assumption that $\sqrt{7}$ is a rational number is incorrect since it leads to a contradiction.

Therefore $\sqrt{7}$ can't be a rational number, hence it is an irrational number.

Thus, we come across on numbers which are not rational numbers. These are called IRRATIONAL numbers.

SET OF REAL NUMBERS

The set containing rational numbers as well as irrational numbers are called set of real numbers (denoted by R).

Thus $\qquad R = Q \cup Q'$

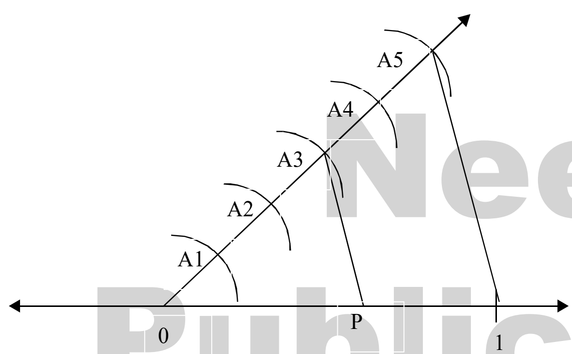
Also intersection of $Q \cap Q' = \phi$ a null set.

THE REAL LINE

You must all be familiar with this notion of a number line. To every point on the number line a number is associated either rational or irrational. Also for every real number there is a unique point on the real number line.

Example 1. Locate the position of $\frac{3}{5}$ on the number line.

Sol. To find the location of $\frac{3}{5}$ on the real number line we divide one unit on the number line into 5 parts and take 3 out of these. Divide the distance 0 to 1 on the number line into 5 parts as shown in the diagram below, A set of irrational numbers is denoted by I



P is the point on the number line which represents the real number $\frac{3}{5}$.

COMPLEX NUMBERS

The equation $x^2 + 5 = 0$ gives the solution that $x = \sqrt{-5}$ which is not a real number therefore this equation has no solution in the real number system.

To deal with this problem we introduce the concept of *i*.

$$\sqrt{-5} = \sqrt{5} \times \sqrt{-1}$$

$$\Rightarrow \quad \sqrt{-5} = i \times \sqrt{5}$$

$$\Rightarrow \quad \sqrt{-5} = i\sqrt{5}$$

Here $\quad i = \sqrt{-1}$

or $\quad i^2 = -1$

This gives rise to a new set of numbers which are called complex numbers.

Set of Complex Numbers

Set of complex numbers denoted by C is defined as C = $\{a + ib$ / where $a, b \in$ R, $i = \sqrt{-1}\ \}$

Addition and Multiplication of Complex Numbers

If $\quad z = a + ib$

$\quad y = c + id$

Then $\quad z + y = (a + c) + i\,(d + b)$

And $\quad z \times y = (a \times c - b \times d) + i(b \times c + a \times d)$

Every point on a number line represents a real number, while any point on a plane represents a complex number.

Algebraic and Transcendental Numbers

Any number x which is the root of a polynomial equation of the form

$$a_0 + a_1x + a_2x^2 + \ldots + a_{n-1}x^{n-1} + a_nx^n = 0$$

is called an algebraic number. Numbers, which are not algebraic numbers, are called transcendental numbers. π and e are examples of transcendental numbers.

PRINCIPLE OF MATHEMATICAL INDUCTION

Suppose there is a given statement $p(n)$ involving natural number n such that

1. The statement is true for $n = 1$ i.e. P (1) is true
2. If the statement is true for $n = k$ (where k is some +ve integer) then the statement is also true for $n = k + 1$ i.e. truth of $p(k) \rightarrow$ truth of P$(k + 1)$ then $p(n)$ is true for all number n.

Example 1: Prove that $2^n > n$ for all +ve integers *n*.

Sol. Let P(n): $2^n > n$

When $n = 1$, $2^1 > 1$ is true

Therefore P (1) is true

$$\Rightarrow \quad 2^k > k \qquad \ldots(1)$$

Then we shall prove that P$(k + 1)$ is true

$$2^{k+1} = 2^k * 2 > 2 * k \quad \text{..........using (1)}$$

$$\therefore \quad 2^{k+1} > k + k > k + 1$$

$$\Rightarrow \quad 2^{k+1} > k + 1$$

$$\Rightarrow \quad p(k + 1) \text{ is true}$$