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MATHEMATICAL METHODS IN PHYSICS - I

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MATHEMATICAL METHODS IN PHYSICS - I

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QUESTION PAPER

(June – 2017)

(Solved)

MATHEMATICAL METHODS IN PHYSICS-I

Time: 1½ Hours] *[Maximum Marks: 25*

Note: *Attempt all Questions. All questions carry equal marks.*

Attempt any *three* parts:

Q. 1. *(a)* Determine the volume of a parallelepiped whose three adjacent sides are given by $\vec{a} = 2\hat{i} - 4\hat{k}$, $\vec{b} = \hat{i} + 2\hat{j} - \hat{k}$ and $\vec{c} = 2\hat{i} - 3\hat{j} + 4\hat{k}$.

Ans. $\vec{a} = 2\hat{i} - 4\hat{k} \Rightarrow a = (2, 0, -4)$

$\vec{b} = \hat{i} - 2\hat{i} - \hat{k} \Rightarrow b = (1, 2, -1)$

$\vec{c} = 2\hat{i} - 3\hat{i} + 4\hat{k} \Rightarrow c = (2, -3, 4)$

Volume of parallelepiped

$$V = |a.\,(b \times c)|$$

$$V = (2, 0, -4).\begin{vmatrix} i & j & k \\ 1 & 2 & -1 \\ 2 & -3 & 4 \end{vmatrix}$$

$$= (2,0,-4).\begin{vmatrix} 2 & -1 \\ -3 & 4 \end{vmatrix} i - \begin{vmatrix} 1 & -1 \\ 2 & 4 \end{vmatrix} j + \begin{vmatrix} 1 & 2 \\ 2 & -3 \end{vmatrix} k$$

$= (2, 0, -4) . (8 - 3)\, i - (4 + 2)\, i + (-3 - 4)\, k$

$= (2, 0, -4). (5, 6, -7)$

$= (10 + 28)$

$V = 38$ **Ans.**

***(b)* A rigid body is rotating with an angular speed of 3-0 rads^{-1} about an axis OL = $2\hat{i} - 2\hat{j} + \hat{k}$ where O is the origin. Determine the velocity of the body at trhe point (4, 1, 2).**

Sol. Angular speed $= 30$ rad s^{-1}

$$OL = 2\hat{i} + 2\hat{j} + \hat{k}$$

$P(4, 1, 2) = \Rightarrow 4i + j + 2k$

$r = (4i + j + 2k) - (2i - 2j + k)$

$= 2i + 3j - k$

Henec if v is velocity of pt. P

We have

$v = w$ cross r

$= z \times (2i + 3j + k)$

$= 6i + 9j - 3k$

***(c)* A particle moves along a curve whose parametric equations are $x = 3t^2$, $y = t^2 - 2t$, $z = t^3$, where the parameter t is time. Calculate its velocity and acceleration at $t = 2$ s.**

Ans. Given data

Particle moves along the curve

$x = 3t^2$, $y = t^2 - 2$, $z = t^3$

Where t = time

velocity of particle at time t = 2s.

is, $\dfrac{d\vec{r}}{dt}$ where $\vec{r} = r\hat{i} + y\hat{j} + z\hat{k}$

$= 3t^2\hat{i} + (t^2 - 2t)\hat{j} + t^3\hat{k}$

$\Rightarrow \dfrac{d}{dt}\left[3t^2\hat{i} + (t^2 - 2t)\hat{j} + t^3\hat{k}\right]$

Velocity $\Rightarrow 6t\,\hat{i} + (2t - 2)\,\hat{j} + 3t^2\hat{k}$

Acceleration $\Rightarrow \dfrac{d^2\vec{r}}{dt^2} \Rightarrow 6\hat{i} + 2j + 6t\hat{k}$

$|\overline{V}| = \sqrt{36t^2 + (4t^2 - 8t + 4) + 9t^4}$

$\Rightarrow \sqrt{40t^2 - 3t + 4 + 9t^4}$

$= \sqrt{40 \times 4 - 8 \times 2 + 4 + 9 \times 16}$

$= \sqrt{160 - 16 + 4 + 144}$

$= \sqrt{144 + 4 + 144}$

$\Rightarrow = \sqrt{292}$

$= 17.088$ **Ans.**

$$|\bar{a}| = \sqrt{36+4+36t^2}$$
$$= \sqrt{40+36\times 4}$$
$$= \sqrt{40+144}$$
$$\Rightarrow \quad = \sqrt{184}$$
$$\Rightarrow \quad = 13.5646$$ **Ans.**

***(d)* Calculate a unit vector normal to the surface $x^3 + y^3 + 3xyz = 3$ at the point (1, 2, –1).**

Sol. $f(x, y, z) = x^3 + y^3 + 3xyz - 3 = 0$

The graidient of $f(x, y, z)$ at point x, y, z is a vector normal to the surface at this point.

The gradient is obtained as follows

$\nabla f(x, y, z) = (f_x, f_y, f_z) = 3(x^2 + yz, y^2 + xz, xy)$

at point (1, 2, –1) has the value 3 (– 1, 3, 2) and the unit vector is

$$\frac{\{-1,3,2\}}{\sqrt{1+3^2+2^2}} = \left\{1\frac{1}{\sqrt{14}}, \frac{3}{\sqrt{14}}, \sqrt{\frac{2}{7}}\right\}$$

***(e)* Prove that:**

$$\nabla^2 (r^n) = n(n+1)\, r^{n-2}$$

Sol. $\nabla^2 (r^n) = n(n+1)\, r^{n-2}$

We have $r = \sqrt{x^2+y^2+z^2}$

So we have $\frac{\partial (r^n)}{\partial x} = nxr^{n-2}$

Therefore

$$\frac{\partial (r^n)}{\partial x^2} = nr^{n-2} + n\,x\,(n-2)x\, r^{n-4}$$
$$= nr^{n-4}(r^2 + (n-2)\,x^2)$$

8 because of the symmetry in r w.r.t. x, y, & z, we also have

$$\frac{\partial (r^n)}{\partial y^2} = nr^{n-4}(r^2 + (n-2)y^2)$$

$$\frac{\partial (r^n)}{\partial z^2} = nr^{n-4}(r^2 + (n-2)z^2)$$

Taking the sum of these we get

$$\nabla^2 (r^n) = nr^{n-4}(3r^2 + (n-2)(x^2+y^2+z^2))$$

$$\nabla^2 (r^n) = n(n+1)r^{n-2}$$ Hence, proved

Q. 2. Using the line integral, calculate the work done by the force $\vec{F} = (2y+3)\,\hat{i} + xz\,\hat{j} + (yz - x)\hat{k}$ when it moves a particle from the point (0, 0, 0) to the point (2, 1, 1) along the curve $x = 2t^2, y = t, z = t^3$.

Sol. $\vec{F} = (2y+3)\,\hat{i} + xz\,\hat{j} + (yz-x)\hat{k}$

$x = 2t^2$

$y = t$

$z = t^3$

$F = (2t+3)\,\hat{i} + 2t^2.t^3\,\hat{i} + (t.t^3 - 2t^2)\,\hat{k}$

$$\frac{d\vec{r}}{dt} = 4t\,\hat{i} + \hat{j} + 3t^2\,\hat{k}$$

$$F.\frac{d\vec{r}}{dt} = \left((2t+3)\hat{i} + 2t^5 j + (t^4 - 2t^2)\hat{k}\right)$$
$$\left(4t\,\hat{i} + \hat{j} + 3t^2 - \hat{k}\right)$$

$$\Rightarrow \left[((2t+3).4t) + 2t^5.1 + (t^4 - 2t^2).3t^2\right]$$

$$\Rightarrow 3t^2 + 12t + 2t^5 + 3t^6 - 6t^4$$

$$\int \vec{F}\,d\vec{r} = \int \left(8t^2 + 12t + 2t^5 - 3t^6 - 6t^4\right)dt$$

$$\Rightarrow \frac{-3x^7}{7} + \frac{x^6}{3} - \frac{6x^5}{5} + \frac{8x^3}{3} + 6x^2 + \text{ content–}$$

Or

Using Stokes, theorem, evaluate the integral $\int_C \vec{A}.d\vec{l}$, where $\vec{A} = z^2\hat{j} + yz\hat{k}$ and C is a closed path in the yz-plane joining the points O (0, 0, 0), P (0, 3, 0) and Q (0, 3, 1).

Ans. Using Stokes theorem–

$$\int_C \vec{A}\,.\,\vec{dl}$$

$$\vec{A} = z^2\,\hat{j} + yz\hat{k} \Rightarrow dx\,\hat{i} + dy\,\hat{j} + dz\,\hat{k}$$

$\therefore$ A. $dl = z^2dy + yzdz$

Hence,

$$\oint_C \text{A.dl} = \int_O^P \text{A. dl} + \int_P^C \text{A.dl} + \int_Q^O \text{A.dl}$$

For O to P, $x = 0$, y goes from o to 3, $z = 0$

Thus $\int_O^P$ A.dl = 0

Sample Preview
of
The Chapter

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MATHEMATICAL METHODS IN PHYSICS-I

VECTOR CALCULUS

Vector Algebra

1

INTRODUCTION

In this chapter, we study vector algebra. We start with basic definition and elementary operations on vectors, such as addition and subtraction of vectors, multiplication of a vector by a scalar, etc. This is followed by, representation of vectors in component forms. Next come vector products. Geometrical and physical interpretations are provided everywhere to make the study more meaningful.

We cover following topics in this chapter:

1. Scalars and vectors,
2. Definitions of a null (or zero) vector, a unit vector, negative of a vector and equality of vectors,
3. Coinitial vectors, like and unlike vectors, free and localized vectors, coplanar and co-linear vectors,
4. Addition and subtraction of vectors, both graphically and analytically,
5. Vector multiplication by a scalar,
6. Systems of linearly independent and dependent vectors,
7. Scalar and vector products of two vectors and their geometrical interpretation,
8. Multiple products of vectors,
9. Scalar triple products and vector triple products and their geometrical interpretation. These are often used in physics applications related to electric and magnetic phenomena,
10. Quadruple product of vectors,
11. Polar and Axial vectors, and
12. Applications of vector algebra,

CHAPTER AT A GLANCE

VECTOR ALGEBRA, THE WAY YOU LEARNT IT AT SCHOOL

Physical quantities like mass of a body, charge of an electron, specific heat of water, resistance of a resistor, diameter of a circle and volume of a cube can be expressed by a single number in appropriate unit of measure. Any such quantity is called a scalar. Length, temperature, time, density, frequency are other scalar quantities.

Notation

Vectors are written in bold face letters e.g. ***v***, ***a***, **F**, etc. Vectors are denoted by drawing arrows above them e.g. $\vec{v}$, $\vec{a}$, $\vec{F}$ or by drawing a line (straight or curly) below them e.g. $\underline{v}$, $\underline{a}$, $\underline{F}$ or $\underset{\sim}{v}$, $\underset{\sim}{a}$, $\underset{\sim}{F}$. The magnitude of vector v is denoted by $|v|$. It is called modulus of v. It may be denoted by light letter in italics. In diagrams vectors are shown as straight lines with arrowheads on them. A directed line segment AB is called as AB.

Equality of Vectors: Two vectors are said to be equal if they have

(*a*) same length

(*b*) same or parallel supports

(*c*) same sense.

Addition of Vectors

The idea of addition of two vectors comes from displacements. Also, it was found that resultant of two forces can be determined by the parallelogram law or by the triangle law.

Let ***a*** and ***b*** be two vectors. Also, let vector ***a*** be the directed line segment AB and the vector ***b*** be the directed segment BC (terminal point B of ***a*** is thus the initial point of ***b***) (Fig. 1.7). Then the directed line segment AC (i.e., AC) represents the sum (or resultant) of ***a*** and ***b*** and is denoted as ***a*** + ***b***.

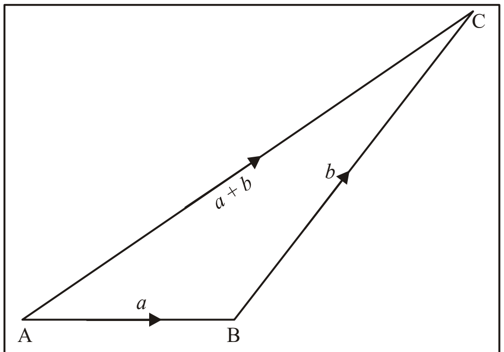


Fig. 1.7. Addition of two Vectors

Clearly, $\overrightarrow{AC} = \overrightarrow{AB} + \overrightarrow{BC} = \vec{a} + \vec{b}$.

This method of drawing a triangle to define the vector sum ($a + b$) is called triangle law of addition of two vectors. The law is stated as follows:

If two vectors are represented by two sides of a triangle, taken in order, then their sum (or resultant) is represented by the third side of the triangle taken in the reverse order.

As any side of a triangle is less than the sum of the other two sides of the triangle; the modulus of AC is less than the sum of moduli of AB and BC.

1. **Vector addition is commutative:** If ***a*** and ***b*** are any two vectors, then
   a* + *b* = *b* + *a

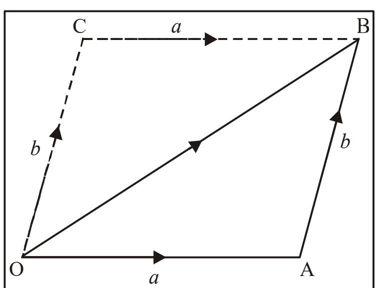


Fig. 1.8. Vector addition is commutative

Let OA = ***a*** and AB = ***b*** (Figure 1.8)

∴ OB = OA + AB

= ***a*** + ***b*** ... (1)

In parallelogram OABC. Then OC = AB = ***b*** and CB = OA = ***a***

∴ OB = OC + CB

= $b + a$ ...(2)

From equations (1) and (2), we obtain,

a* + *b* = *b* + *a

2. **Vector addition is associative:** If ***a, b, c*** are any three vectors, then
   a* + (*b* + *c*) = (*a* + *b*) + *c

This equation is easily verified from Figure 1.9.

The sum of three vectors ***a, b, c*** is independent of the order in which they are added and is written as:

***a* + *b* + *c*.**

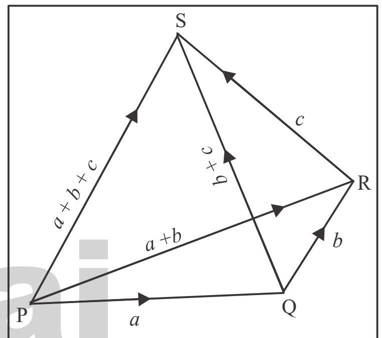


Fig. 1.9. Vector Addition is associative

3. **Existence of Additive Identity:** For any vector ***a***,
   ***a* + 0 = *a* = 0 + *a*,**
   where **0** is a null (or zero) vector.
   Here, **0** is called additive identityof vector addition.
4. **Existence of Additive Inverse:** For any vector ***a***, there exists another vector – a is such that
   ***a* + (– *a*) = 0**

Here, **–*a*** is a vector of length |a| and direction opposite to that of ***a***.

The vector **(– *a*)** is called additive inverse of vector a.

Multiplication of a Vector by a Scalar

(I) m(***a*** + ***b***) = m***a*** + m***b***

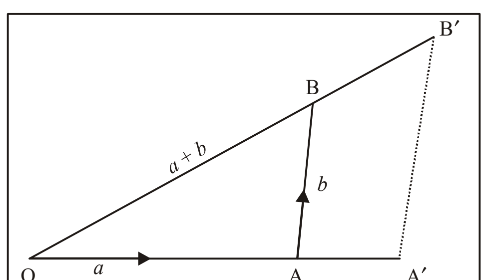


Fig. 1.12

Here, OA′ = $m\boldsymbol{a}$ and A′B′ = $m\boldsymbol{b}$

∴ OB′ = $m\boldsymbol{a} + m\boldsymbol{b}$

Also OB′ = $m(\boldsymbol{a} + \boldsymbol{b})$

(II) $(m + n)\boldsymbol{a} = m\boldsymbol{a} + n\boldsymbol{a}$ (Distributive law)

(III) $m(n\,\boldsymbol{a}) = (mn)\,\boldsymbol{a} = mn\boldsymbol{a}$ (Associative Law)

(IV) $1\boldsymbol{a} = \boldsymbol{a}$ (Existence of Multiplicative Identity)

(V) $0\boldsymbol{a} = 0$

(VI) $(-1)\,\boldsymbol{a} = -\boldsymbol{a}$

If $\hat{a}$ is a unit vector having the same direction as $\boldsymbol{a}$, then

$$\boldsymbol{a} = |a|\hat{a} \text{ and } \hat{a} = \frac{\boldsymbol{a}}{|a|}.$$

Subtraction of Vectors

The difference $\boldsymbol{a} - \boldsymbol{b}$ of two vectors is defined as the sum $\boldsymbol{a} + (-\boldsymbol{b})$, where $(-\boldsymbol{b})$ is the negative of b. Geometrically the difference $\boldsymbol{a} - \boldsymbol{b}$ is as shown in Fig. 1.13.

It is obvious that:

$$\boldsymbol{a} - \boldsymbol{a} = \boldsymbol{0}$$

and $$\boldsymbol{a} + \boldsymbol{0} = \boldsymbol{a}.$$

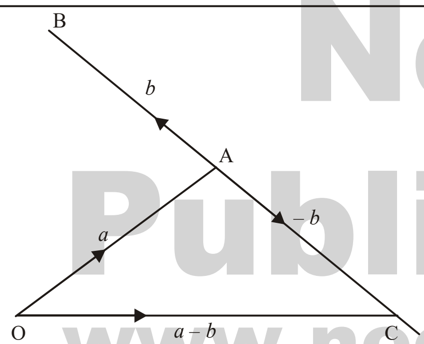


Fig. 1.13. Difference of two Vectors

When two vectors are given in component forms their difference is obtained by subtracting the vectors component wise.

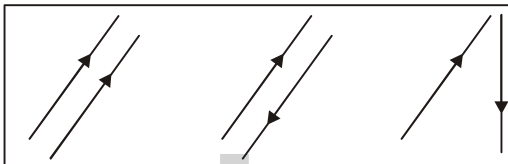
Thus, if $\boldsymbol{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$ and

$\boldsymbol{b} = b_1\hat{i} + b_2\hat{j} + b_3\hat{k}$, then

$$\boldsymbol{a} - \boldsymbol{b} = (a_1 - b_1)\hat{i} + (a_2 - b_2)\hat{j} + (a_3 - b_3)\hat{k}$$

Types of Vectors

1. **Zero Vector or Null Vector:** A vector having length as zero is called a Null or Zero Vector and is denoted by 0.
   Obviously, a null vector has no direction.
   Hence, a = AB is a null vector if and only if $|a|$= 0, i.e. if and only if |AB|= 0, i.e., if and only if A and B coincide
   Any non-zero vector is termed as a proper vector.
2. **Unit Vector:** A vector whose length (modulus or magnitude) is unity is called a unit vector. Usually, a unit vector is denoted by a single letter with a cap '$\hat{a}$' over it.
   Hence, $\hat{a}$ denotes a unit vector.
   Also $a = |a|\hat{a}$ or $\hat{a} = \dfrac{\hat{a}}{|a|}$.
3. **Co-initial Vectors:** All vectors with the same initial point are called co-initial vectors. Clearly, AB, AC, AD are all co-initial vectors.
4. **Like and Unlike Vectors:** Vectors are said to be like if they have the same direction and unlike if they have opposite directions. (See Figure 1.4)



(a) Like Vectors *(b)* Unlike Vectors *(c)* Vectors with different directions

Fig. 1.4

Both unlike and like vectors have the same line of action or have the lines of action parallel to one another; such vectors are also termed Collinear or Parallel Vectors.

Multiplication of Vectors

After a vector is multiplied by another vector, the outcome can be scalar or a vector. There are two different ways of multiplying vectors. First is the scalar or dot or inner product which is just a number (or scalar) of magnitude alone. The other is vector or cross product, which is a vector with a definite direction.

1. Scalar or Dot Product : The scalar or dot or inner product of two vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ in the three-dimensional space is written as $\boldsymbol{a}.\,\boldsymbol{b}$(read as '**a dot b**') and is defined as:

$$\boldsymbol{a.b} = \begin{bmatrix} |a||b|\cos\gamma, & \text{when } \boldsymbol{a} \neq 0, \boldsymbol{b} \neq 0 \\ 0 & \text{when } \boldsymbol{a} = 0 \text{ or } \boldsymbol{b} = 0 \end{bmatrix}, \propto$$

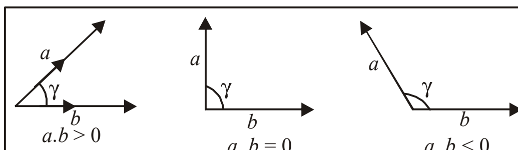


Fig. 1.15, Angle between Vectors and their dot product.

where $\gamma(0 \leq \gamma \leq \pi)$ is the angle between a and b (computed when the vectors have their initial points co-inciding) (Fig. 1.15).

The value of the dot product is a scalar (a real number), and hence the term "scalar product". The cosine in equation may be positive, zero or negative. That is so also for the dot product.

Angle γ in the equation lies between 0 and π and since $\cos \gamma = 0$, if $\gamma = \pi/2$, an important result follows:

Two non-zero vectors are orthogonal (perpendicular) if and only if their dot product is zero.

If $\boldsymbol{b} = \boldsymbol{a}$ in the equation then $a.a = \left|a^2\right|$. This means the length (or Euclinear norm or modulus or magnitude) of a vector can be expressed in terms of scalar product as

$$|a| = \sqrt{\boldsymbol{a.a}} \ (\geq 0). \ ....(a)$$

These relations and give the angle γ between two non-zero vectors as:

$$\cos \gamma = \frac{\boldsymbol{a.b}}{|a||b|} = \frac{\boldsymbol{a.b}}{\sqrt{a.a}\sqrt{b.b}}$$

The scalar product has the following properties:

P(1) $(q_1\boldsymbol{a} + q_2\boldsymbol{b}).c = q_1\boldsymbol{a}.\ \boldsymbol{c} + q_2\boldsymbol{b}.\boldsymbol{c}$ (linearity)

P(2) $\boldsymbol{a}.\boldsymbol{b} = \boldsymbol{b}.\boldsymbol{a}$ (symmetry or commutative law)

P(3) $a.\boldsymbol{a} \geq 0$

Also $a.\boldsymbol{a} = 0$ if and only if $\boldsymbol{a} = 0$

(positive definiteness)

In property P(1) putting $q_1 = 1$ and $q_2 = 1$, we get

P(4) $(\boldsymbol{a} + \boldsymbol{b}).\boldsymbol{c} = \boldsymbol{a}.\boldsymbol{c} + \boldsymbol{b}.\boldsymbol{c}$ (distributivity)

Thus, scalar product is commutative and distributive with respect to vector addition.

The definition of scalar product, also means

P(5) $|\boldsymbol{a}.\boldsymbol{b}| \leq |\boldsymbol{a}||\boldsymbol{b}|$ $(\because |\cos \gamma| \leq 1)$ (Schwarz Inequality)

Using the definition and simplifying, we obtain another property as:

P(6) $|\boldsymbol{a}+\boldsymbol{b}|^2 + |\boldsymbol{a}-\boldsymbol{b}|^2 = 2\left(|\boldsymbol{a}|^2 + |\boldsymbol{b}|^2\right)$

(Parallelogram equality)

Moreover, if $\hat{i}, \hat{j}, \hat{k}$ are unit vectors forming an orthogonal trial, then, from definition of scalar product, We have

P(7) If $\boldsymbol{a}\ .\ \boldsymbol{b} = \boldsymbol{0}$ and $\boldsymbol{a}, \boldsymbol{b}$ are non-zero vectors then a and bc are perpendicular to each other.

P(8) $\sqrt{\boldsymbol{a.a}} = |a|$

P(9) $\begin{Bmatrix} \hat{i}.\hat{i} = 1, & \hat{j}.\hat{j} = 1, & \hat{k}.\hat{k} = 1 \\ \hat{i}.\hat{j} = 0 & \hat{j}.\hat{k} = 0, & \hat{k}.\hat{i} = 0 \end{Bmatrix}$

If vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ are represented in terms of components, say,

$$\boldsymbol{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k} \text{ and } \boldsymbol{b} = b_1\hat{i} + b_2\hat{j} + b_3\hat{k},$$

then their scalar product is equal to

$$\boldsymbol{a.b} = a_1b_1 + a_2b_2 + a_3b_3 \qquad \text{(by P7)}$$

and

$$\cos \gamma = \left(\frac{\boldsymbol{a.b}}{|a||b|}\right) = \frac{(a_1b_1 + a_2b_2 + a_3b_3)}{\left(\sqrt{a_1^2 + a_2^2 + a_3^2}\sqrt{b_1^2 + b_2^2 + b_3^2}\right)},$$

where γ is the angle between a and b.

NEW LOOK AT VECTORS

Vector Components Relative to a Coordinate System: The regular coordinate system in space has axes which are mutually perpendicular straight lines. On these three axes, same scale is used. Hence, three unit points on the axes, with coordinates (1, 0, 0), (0, 1, 0) and (0, 0, 1) will have the same distance from the origin, the point of intersection of the axes. This rectangular coordinate system is known as Cartesian coordinate system in space [Fig. 1.5(a)].

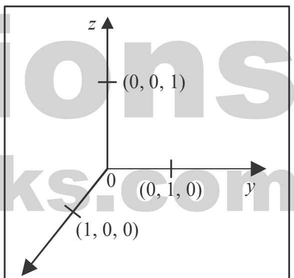


Fig. 1.5.(a) **Cartesian Coordinate System**

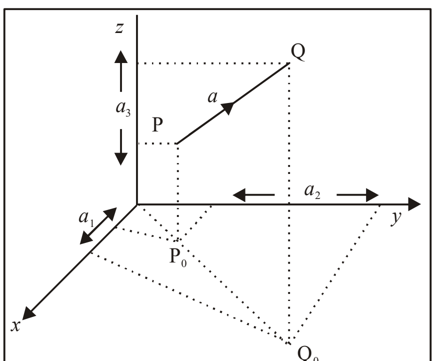


Fig. 1.5. *(b)* Components of a Vector