Basic Mathematics

By: Ranveer

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BASIC MATHEMATICS

ALGEBRA-I

Determinants

1

INTRODUCTION

In algebra, the determinant is a special number associated with any square matrix. The fundamental geometric meaning of a determinant is a scale factor for measure when the matrix is regarded as a linear transformation. Also, we can say this is the mathematical objects that are very useful in the analysis and solution of systems of linear equations.

The linear equation is an algebraic equation in which each term is either a constant or the product of a constant and a single variable. Linear equations can have one or more variables. Various elementary methods are used to solve these linear equations which involve two or three variables. But these methods are not helpful where a large number of equations involve more than three variables. Thus, few other methods are involved for such equations; such as Matrices and Determinants. By the end of this chapter, you will be able to define a matrix and a determinant, addition and multiplication of two matrix, obtain the determinant of a matrix, and compute the inverse of a matrix, Matrices and Determinants.

CHAPTER AT A GLANCE

DEFINITION

We define the determinant function

det : $M_n(F) \to F$ by induction on n.

When $n = 1$, det A = det $[a] = a$

When $n = 2$, det A = det $\begin{bmatrix} a & b \\ c & d \end{bmatrix} = ad - bc$

When $n = 3$,

$$\det A = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

$$= a_{11} \det \begin{pmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{pmatrix} - a_{12} \det \begin{pmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{pmatrix} + a_{13} \det \begin{pmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{pmatrix}$$

Determinant

In double suffix notation, a determinant of order n is defined as

$$\Delta = \begin{vmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \dots & \dots & \dots & \dots \\ a_{n1} & a_{n2} & a_{n3} & a_{nn} \end{vmatrix}$$

It consists of n rows and n columns. The element a_{11}, a_{12} .... can be real or complex or fraction. The element a_{ij} belongs to the ith row and jth column. The elements a_{11}, a_{22}, a_{33}, ..., a_{nn} consisting the leading diagonal or principal diagonal of the determinant Δ.

Note: A determinant has definite value.

Determinant of a matrix: For any square matrix of order 2, we have found a necessary and sufficient condition for invertibility. Indeed, consider the matrix

The matrix A is invertible if and only if . We called this number the determinant of A. It is clear from this, that we would like to have a similar result for bigger matrices (meaning higher orders). So is there a similar notion of determinant for any square matrix, which determines whether a square matrix is invertible or not?

In order to generalize such notion to higher orders, we will need to study the determinant and see what kind of properties it satisfies. First let us use the following notation for the determinant.

General Formula for the Determinant Let A be a square matrix of order n. Write $A = (a_{ij})$, where aij is the entry on the row number i and the column number j, for $i = 1, ..., n$ and $j = 1,... n$. For any i and j, set A_{ij} (called **the cofactors**) to be the determinant of the square matrix of order $(n - 1)$ obtained from A by removing the row number i and the column number j multiplied by $(-1)i + j$. We have

$$\det(A) = \sum_{j=1}^{j=n} a_{ij} A_{ij}$$

for any fixed i, and

$$\det(A) = \sum_{i=1}^{i=n} a_{ij} A_{ij}$$

for any fixed j. In other words, we have two type of formulas: along a row (number i) or along a column (number j). Any row or any column will do. The trick is to use a row or a column which has a lot of zeros.

In particular, we have along the rows

$$\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & k \end{vmatrix} = a\begin{vmatrix} e & f \\ h & k \end{vmatrix} - b\begin{vmatrix} d & f \\ g & k \end{vmatrix} + c\begin{vmatrix} d & e \\ g & h \end{vmatrix}$$

or

$$\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & k \end{vmatrix} = -d\begin{vmatrix} b & c \\ h & k \end{vmatrix} + e\begin{vmatrix} a & c \\ g & k \end{vmatrix} - f\begin{vmatrix} a & b \\ g & h \end{vmatrix}$$

or

$$\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & k \end{vmatrix} = g\begin{vmatrix} b & c \\ e & f \end{vmatrix} + h\begin{vmatrix} a & c \\ d & f \end{vmatrix} + k\begin{vmatrix} a & b \\ d & e \end{vmatrix}$$

As an exercise write the formulas along the columns.

Minors

If we delete the ith row and jth column in the determinant D, we get another determinant of $(n - 1)$th order called minor of the element a_{ij}.

Let us consider a determinant D_1 of third order,

$$D_1 = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

Let the minors of the elements $a_{11}, a_{12}, a_{13} \cdots a_{33}$ be denoted by $M_{11}, M_{12} \cdots$ respectively, then

$$M_{11} = \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix}, \; M_{12} = \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} \text{ etc.}$$

Co-factors

The co-factor of an element a_{ij} of a determinant Δ is denoted by A_{ij} and is defined by $A_{ij} = (-1)^{i+j} M_{ij}$.

Cofactors

$c_{ij} = (-1)^{i+j} M_{ij}$ for any minor M_{ij} of a_{ij} element. Now we find cofactor of every element of a $m \times m$ matrix, and put in matrix form, viz.,

$$C = \begin{pmatrix} c_{11} & c_{12} & ... & c_{1m} \\ c_{21} & c_{22} & ... & c_{2m} \\ ... & ... & ... & ... \\ c_{m1} & c_{m2} & ... & c_{mm} \end{pmatrix},$$

known as cofactor matrix, where $(i, j)^{th}$ vector is deleted.

Now adjoin of A, written as Adj A is given by

$$\text{Adj A} = \text{Transpose of cofactor matrix}$$
$$= C^t$$

then $$A^{-1} = \frac{1}{|A|} \text{Adj A}$$

where Adj A is the inverse of matrix A, with condition that $|A| \neq 0$.

We can easily see

$$A.[\text{Adj}(A)] = [\text{Adj}(A)]A = \det(A).I$$

A system of linear equations

$$a_{11}x_1 + a_{12}x_2 + ... + a_{1n}x_n = b_1$$
$$a_{21}x_1 + a_{22}x_2 + ... + a_{2n}x_n = b_2$$
....
....

$a_{n1}x_1 + an_2x_2 + ... + a_{nn}x_n = b_n$, can be solved as follows:

Let $$A = (a_{ij})_{n\times n}, X = \begin{pmatrix} x_1 \\ x_2 \\ x_n \end{pmatrix}, B = \begin{pmatrix} b_1 \\ b_2 \\ b_n \end{pmatrix}$$

then $AX = B$

$\Rightarrow$ $X = A^{-1}B$.

Theorem 1: Let A = [a_{ij}] then

(a) **$a_{11}c_{11} + a_{12}c_{12} + ... + C_n$ = det (A)**

(b) **$a_{ji}C_{ji} + a_{j2}C_{j2} + ... + a_{jn}C_{jn}$ = Det (A).**

Hint: For Self Attempt.

Theorem 2: Let A be an $n \times n$ matrix over F, Then

A. (Adj(A)) = (Adj(A)).A = det (A)I.

Proof: Recall matrix multiplication from Unit 7.

Now

$$A(\text{Adj}(A)) = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix} \begin{bmatrix} C_{11} & C_{21} & \cdots & C_{1n} \\ C_{12} & C_{22} & \cdots & C_{n2} \\ \vdots & \vdots & & \vdots \\ C_{1n} & C_{2n} & \cdots & C_{nn} \end{bmatrix}$$

By Theorem 1 we know that $a_{il}C_{il} + a_{i2} C_{i2} + ...+ a_{in}C_{in} = \det(A)$, and $a_{i1}C_{j1} + a_{i2}C_{j2} + ...+ a_{in}C_{jn} = 0$ if $i \neq j$. Therefore,

$$A(\text{A}dj(A)) = \begin{bmatrix} \det(A) & 0 & \cdots & 0 \\ 0 & \det(A) & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ : & : & \cdots & : \\ 0 & 0 & \cdots & \det(A) \end{bmatrix}$$

$$\det(A) = \begin{bmatrix} 1 & 0 & ... & 0 \\ 0 & 1 & ... & 0 \\ 0 & 0 & ... & 0 \\ : & : & ... & : \\ 0 & 0 & ... & 1 \end{bmatrix} \det(A)I.$$

Theorem 3: Let the matrix equation of a system of linear equations be

$$AX = B, \text{ where } A = [a_{ij}]n \times n, X = \begin{bmatrix} x_1 \\ . \\ . \\ x_n \end{bmatrix} B = \begin{bmatrix} b_1 \\ . \\ . \\ b_n \end{bmatrix}$$

Let the columns of A be $C_1, C_2,......C_n$. If $\det(A) \neq 0$, the given system has a unique solution, namely,

$x_1 = D_1/D, ..., x_n = D_n/D$, where

$D_i = \det(C_1, ... C_{i-1}, B., C_{i+1}, ..., C_n)$

= determinant of the matrix obtained from A by replacing the ith column of B, and D = det (A).

Now let us see what happens if B = 0. As we know that AX = 0 has $n - r$ linearly independent solutions, where r = rank A. The following theorem tells this condition in terms of det(A).

Theorem 4: The homogeneous system AX = 0 has a non-trivial solution if and only if. det (A) = 0.

Proof: First assume that AX = 0 has a non-trivial solution. Suppose, if possible, that $\det(A) \neq 0$. Then Cramer's Rule says that AX = 0 has only the trivial solution X = 0 (because each $D_i = 0$ in Theorem 3). This is a contraction to our assumption. Therefore, det (A) = 0.

Conversely, if det(A) = 0. then A is not invertible. $\therefore$, the linear mapping $A : V_n(F) \to V_n(F) : A(X) = AX$ is not invertible. $\therefore$, this mapping is not one-one.

Therefore, Ker A $\neq$ 0 that is AX = 0 for some non-zero $X \in V_n(F)$. Thus, AX = 0 has a non-trivial solution.

Theorem 5: Let $X_1, X_2, ... X_n \in V_n(F)$. Then $X_1, X_2, ... X_n$ are linearly dependent over the field F if and only if $\det(X_1, X_2, ... X_n) = 0$.

Proof: Let $U = (X_1, X_2, ... X_n)$ be the $n \times n$ matrix whose column vectors are $X_1, X_2 .... X_n$. Then $X_1, X_2, ..., X_n$ are linearly dependent over F if and only if there exist scalars $\alpha_1, \alpha_2, ..., \alpha_n \in F$ not all zero, such that $\alpha_1 X_1 + \alpha_2X_2 + ... + \alpha_nX_n = 0$.

$$\text{Now, } U\begin{bmatrix} \alpha_1 \\ \alpha_2 \\ . \\ . \\ \alpha_n \end{bmatrix} = (X_1, X_2, ...X_n)\begin{bmatrix} \alpha_1 \\ \alpha_2 \\ . \\ . \\ \alpha_n \end{bmatrix}$$

$$= X_1\alpha_1 + X_2\alpha_2 + ... + X_n\alpha_n$$
$$= \alpha_1X_1 + \alpha_2X_2 + ... + \alpha_nX_n.$$

Thus, $X_1, X_2, ..., X_n$ are linearly dependent over F if and only if UX = 0 for some non zero

$$X = \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ . \\ \alpha_n \end{bmatrix} \in V_n(F)$$

But this happens if and only if det(U) = 0, by Theorem 4. Thus, Theorem 6 is proved.

Theorem 5 is equivalent to the statement $X_1, X_2, ... X_n \in V_n(F)$ are linearly independent if and only if $\det(X_1, X_2, ... X_n) \neq 0$.

DETERMINANTS OF ORDER 2 AND 3

Determinant of order 2, and 3 are written as:

$$\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{23} \end{vmatrix} \text{ and } \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

Where $a_{ij} \in C \; \forall \; i, j$

Order two

$$\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}a_{22} - a_{12}a_{21}$$

$$\begin{vmatrix} 2 & 3 \\ -1 & 2 \end{vmatrix} = 2.\ 2 - [(-1).\ 3] = 4 + 3 = 7$$

Example:

$$\begin{bmatrix} 3 & 2 \\ 2 & 8 \end{bmatrix} = (3)(8) - (2)(2)$$
$$= 24 - 4 = 20$$

In the second order determinant, we directly multiply diagonal elements.

Order three

Consider an arbitrary 3 × 3 matrix, A = (a_{ij}). The determinant of A is defined as follows:

$$A = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

$a_{11}\, a_{22}\, a_{33} + a_{12}\, a_{23}\, a_{31} + a_{13}\, a_{21}\, a_{32} -$
$- a_{13}\, a_{22}\, a_{31} - a_{12}\, a_{21}\, a_{33} - a_{11}\, a_{23}\, a_{32}.$

Example:

$$\begin{bmatrix} 2 & 3 & 4 \\ 5 & 6 & 7 \\ 1 & -3 & 2 \end{bmatrix}$$

Expanding the determinant along the first row

$$= 2\begin{bmatrix} 5 & 7 \\ -3 & 2 \end{bmatrix} - 3\begin{bmatrix} 6 & 7 \\ 1 & 2 \end{bmatrix} + 4\begin{bmatrix} 6 & 5 \\ 1 & -3 \end{bmatrix}$$
$$= 2\,(10 + 21) - 3\,(12 - 7) + 4\,(-18 -5)$$
$$= 62 - 15 - 92 = -45$$

$$A = \begin{vmatrix} 3 & 2 & 1 \\ 0 & 2 & -5 \\ -2 & 1 & 4 \end{vmatrix}$$
$$= 3 \cdot 2 \cdot 4 + 2 \cdot (-5) \cdot (-2) + 1 \cdot 0 \cdot 1$$
$$- 1 \cdot 2 \cdot (-2) - 2 \cdot 0 \cdot 4 - 3 \cdot (-5) \cdot 1$$
$$= 24 + 20 + 0 - (-4) - 0 - (-15)$$
$$= 44 + 4 + 15 = 63$$

Note that there are six products, each consisting of three elements in the matrix. Three of the products appear with a positive sign (they preserve their sign) and three with a negative sign (they change their sign).

PROPERTIES OF DETERMINANTS: EVALUATION OF DETERMINANTS

(1) The value of a determinant remains unchanged when rows and columns are interchanged.

e.g. $$\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = \begin{vmatrix} a & d & g \\ b & e & h \\ c & f & i \end{vmatrix}$$

(2) If any two successive rows or columns are interchanged, then the determinant is multiplied by (–1).

e.g. $$\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = -\begin{vmatrix} a & b & c \\ g & h & i \\ d & e & f \end{vmatrix}$$

(3) If all the elements of one row or column of a determinant are multiplied by the same number (say λ), the value of the new determinat is λ times the value of the given determinant

e.g. $$\begin{vmatrix} \lambda a_1 & b_1 & c_1 \\ \lambda a_2 & b_2 & c_2 \\ \lambda a_3 & b_3 & c_3 \end{vmatrix} = \lambda \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

(4) If all the elements of a row or column of a determinant are zero, the value of whole determinant is zero.

(5) If any two rows or columns of a determinant are identical, the value of determinant is zero.

(6) In a determinant the sum of the products of the elements of any two row or column with co-factors of the corresponding elements of any other row or column is zero.

(7) If, in a determinant each element in any row or column consists of the sum of two terms, then the determinant can be expressed as the sum of two determinants of the same order.

(8) If the elements of a row of column of a determinant are added *m* times the corresponding elements of another row or column, the value of the determinant thus obtained is equal to the value of the original determinant.

AREA OF TRIANGLES USING DETERMINANTS

Area of triangle whose vertices are (x_1, y_1), (x_2, y_2), (x_3, y_3) is given by

$$\Delta = \frac{1}{2}\left[x_1(y_2 - y_3) - x_2(y_1 - y_3) + x_3(y_1 - y_2)\right] \quad ...(i)$$

Now consider the determinant $\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}$ and expand

$$= x_1\begin{vmatrix} y_2 & 1 \\ y_3 & 1 \end{vmatrix} - x_2\begin{vmatrix} y_1 & 1 \\ y_3 & 1 \end{vmatrix} + x_3\begin{vmatrix} y_1 & 1 \\ y_2 & 1 \end{vmatrix}$$

(expand through the columns)

$$= x_1(y_2 - y_3) - x_2(y_1 - y_3) + x_3(y_1 - y_2) \quad ...(ii)$$

Hence area of a triangle having vertices at (x_1, y_1), (x_2, y_2) and (x_3, y_3) is given by